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MINIMIZING LOSS RATIO IN CAR INSURANCE USING OPTIMAL CONTROL THEORY

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ABSTRACT

In this paper we want to optimize premium and reserve in an optimal control insurance model. A simple parameterisation that represents the insurance market's response to an insurer adopting a pricing strategy determined via optimal control theory is introduced. The criterion is that of minimizing exponential utility. A special feature of our construction is to allow for convergence of the reserves to expected reserve at the end of the planning period. We control the premium in order to minimize the loss ratio and reach to the expected reserve. If the market reacts, then the optimal strategy also changes and the premium may take an increasing (decreasing) shift that will result in more (less) insurance sales. After explaining the model, using numerical examples, one of the parameters is estimated using the statistic science followed by solving the problem using PMP method.

Keywords: *Optimal Control, Premium, Dynamical Systems, Reserve, Optimization.*

INTRODUCTION

In actuarial science, a premium principle equates the cost of a general insurance policy to the moments of the corresponding claim arrival and severity distributions (T. Rolski et al.,1999). In order to make a profit and cover their expenses, insurers add a loading to this cost price. Because many lines of insurance, especially automobile insurances, are highly competitive, the loading critically depends on the price of other insurers to ensure comparable costs of insurance policies. Insurance pricing is therefore an important factor in determining the type of insurance company customers select or change in the next insurance period. According to Taylor's model (G.C. Taylor, 1986), that uses optimal control, the premium is set based on the average insurance market price. The Taylor model is based on, a discrete, demand model for pricing in which the future average market price is exactly evaluated and new and existing premium holders are required to pay the same current premium rate (P. Emms, S. Haberman, 2005).

It is difficult to determine premium price using a discrete deterministic model if the average market premium is a continuous stochastic process. Continuous time models can be sets up in several ways: one can either model the premium rate charged by the insurer for a unit of insurance cover, or charge a premium up front for a finite period of cover thereafter. In the former case, policyholders pay a premium $u(t)$ continuously over the course of their policies (P. Emms, S. Haberman, 2005). It is important point to note is that pricing at a lower rate in the market can result in a negative premium (P. Emms, 2007). This strategy of relatively low initial

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pricing in the market aims to generate sales; by controlling claims, a creditor's insurance can then increase after which he can increase his prices and profits.

Emms and Haberman disucss different modeling constraints (P. Emms, S. Haberman, 2009). All these models assume that there exists a single optimising insurer, whose price does not affect the premiums of other insurers. This however, is applicable to only small insurers in big markets. In mathematical finance, there is a similar supposition for the optimal wealth appropriation problem (R.C. Merton, 1971). It is assumed that the stock rate does not affect the allocation of the investor's stock. Nevertheless, most insurance lines are under the influence of several large insurers who control each other's prices and constantly update their prices. In such markets, competitive pricing models are not responsive and insurers should therefore pay attention to the response of insurers to insurance prices. The competitive pricing model (P. Emms, S. Haberman, 2005) defines the demand function that determines the relationship between premiums and the average premium of the market. If the insurer determines a premium price lower than the average market price, policies will be sold. If the whole insurance market determines a high premium, this would lead to notable sales for an optimising insurer. In reality however, customers do not pay more than the value of the insured, and they maybe at liberty to take or not to take out insurance. In some way, the demand for insurance policies depends on their premiums, and if the price of all policy sellers are far above the cost, our demand law dictates that very few policies will be sold.

In section 2, the motivation behind the modelling is presented and the notation for car insurance pricing is introduced. Section 3 describes how to estimate claim size rate via statistical methods, Section 4, defines the demand functions and solves the model and section 5 summarizes a conclusion.

OPTIMAL CONTROL MODEL PRELIMINARIES

In this section, based on the earlier work of Taylorand Emms (G.C. Taylor, 1986; Paul Emms, 2011), all prices (and claims) per unit of exposure varies according to the insured risk (where the exposure is the unit of risk for an insurer).

- Policyholders pay a premium $u(t)$ continuously over the course of their policies
- We assume $u(t)$ $t \geq 0$ per unit exposure, as a premium (u) at time t for a general insurance policy of fixed duration l.
- We consider $x_1(t)$ per unit exposure as the 'average market premium for a policy' of the same duration.
- $x_2(t)$ is the 'exposure' that expresses a measure of the insurance company's potential liabilities'. It reflects the number of currently in force insurance policies and the potential size of the claims on these policies.
- Suppose that G as the demand function, which is associated with the premium.
- The reserve, $x_3(t)$, represents the amount of current capital held by the insurance company, which increases with the sale of the insurance policy and decreases with payment of claims.
- $x_4(t)$is the claim size rate per unit of exposure.

An insurer who initially tries to gain exposure faces the possibility that the market will follow the same course of action by setting a comparable premium, Consequently, we split up the drift in the market average premium based on the absence or presence of the market's reaction to an optimising insurer. If there is no reactive market, then we will we adopt the expected value principle (Paul Emms, 2011), and assume that the average market premium is directly related to the losses, the average market premium $x_1(t)$ is therefore equal to a constant γ that represents a fixed loss ratio per unit time multiplied by $x_4(t)$is the claim size rate per unit of exposure

$$x_1(t)=\gamma^{-1}x_4(t),$$

If we take $\lambda(u(t)-x_1(t))dt$, to reepresent the change in the average premium market price that follows the market reaction and $\lambda \geq 0$ as the constant reaction of the market to the optimal price of the insurer, the market average premium constraint is therefore computed as follows (Paul Emms, 2011):

$$\dot{x}_1(t)=\gamma^{-1}\dot{x}_4(t)+\lambda(u(t)-x_1(t)) \quad (1)$$

Here, it is assumed that the premium u_t is charged at the beginning of a policy of length $l = \kappa^{-1}$ by the policyholder and each customer who has renewed his insurance policy is considered a new policyholder. Where $x_2(t)$ is the 'exposure' and G is the demand function, which is associated with the premium, we assume that change in exposure over a time interval of length dt is given by (Paul Emms, 2011):

$$\dot{x}_2(t)=x_2(t)(G-k) \quad (2)$$

Equation 2measures the ability of an insurer under its current exposure. Suupposing that large insurers tend to gain greater exposure than small insurers with comparable premiums (Paul Emms, 2011),

- the increase in reserve $x_3(t)$ (i.e. amount of current capital held by the insurance company) from selling insurance at time dt is the increase in exposure from selling policies $x_2(t)$ multiplied by the current premium u (t)

$$x_2(t)\times G\times u(t)\times dt.$$

- the decrease in reserve by paying claims over time dt is the claim size rate $x_4(t)$ per unit of exposure $x_2(t)$

$$x_4(t)\times x_2(t)\times dt \text{ and } x_4(t).$$

Therefore, changes in reserve are as follows (Paul Emms, 2011) where the constant α determines the loss of wealth due to returns to shareholders:

$$\dot{x}_3(t)=-\alpha x_3(t)+x_2(t)\big(Gu(t)-x_4(t)\big) \quad (3)$$

The last equation of state in this model is $x_4(t)$ that denotes the claim size rate of car insurance, so the last constraint of the model (changes in claim size) is as follows (Paul Emms, 2011): Where, μ and σ are constants.

$$\dot{x}_4(t)=x_4(t)\left(\mu+\sigma\dot{x}_3(t)\right) \tag{4}$$

So, $x=(x_1(t),x_2(t),x_3(t),x_4(t))^T$ is the current status vector. Loss ratio in car insurance is $\frac{x_4(t)}{u(t)}$ which is proportional to the amount of losses and received premium. It is important to measure the amount of this deduction for insurance companies especially in the car insurance, as insurance companies are always looking for solutions to reduce the amount of this deduction, because a reduction in this fraction means that the insurance company has reached a low level of risk control. By considering the previous expression, we provide an optimal control in car insurance. For this purpose, based on the desirability of reserve increase at the end of the planning and decrease in loss ratio, we have the following model:

$$\min j[x,u]=(\beta_1\left\|\frac{x_4(t)}{u(t)}\right\|_{L1}+\beta_2\left\|x_d-x_3\left(t_{t_f}\right)\right\|_{L1}) \tag{5}$$

s.t

$\dot{x}_1(t)=\gamma^{-1}\dot{x}_4(t)+\lambda(u(t)-x_1(t))$

$\dot{x}_2(t)=x_2(t)(G-k)$

$\dot{x}_3(t)=-\alpha x_3(t)+x_2(t)(Gu(t)-x_4(t))$

$.\dot{x}_4(t)=x_4(t)(\mu+\sigma\dot{x}_3(t))$

MODIFYING THE MODEL TO A SIMPLER MODEL BASED ON STATISTICAL ESTIMATION

In this section, we use the statistical data of the insurance company to calculate the losses rate $x_4(t)$ to convert the model from fraction to a simpler model and then to solve it in the next section.

For this reason, to provide the losses incurred for three consecutive years from car policy in an insurance company and then calculation of the loss rate in 2012, 2013, 2014, 2015 compared to 4 months from 2012 the year (According to Table1), we conclude that the above statistical data follows the exponential distribution with the distribution function $x_4(t)=1-e^{-0.1375t}$. for fit goodness of the exponential model_ to the existing data from The kolmogorov-Smirnov statistic has been used. The numerical value of this statistic is equal 1.018 With a significant level of 0.463 Is obtained Which indicates that the existing datas at the significant level of 0.05 follow exponential distribution $x_4(t)=1-e^{-0.1375t}$ So, by this assumption and its substitution in model number (5), we get the following model:

$$\min J[x,u]=(\beta_1\left(\left\|\frac{1-e^{-0.1375t}}{u(t)}\right\|_{L1}\right)+\beta_2(\|x_d-x_3(t_f)\|_{L1})) \tag{6}$$

s.t.

$\dot{x}_1(t)=\gamma^{-1}(0.1375\, e^{-0.1375t})+\lambda(\, u(t)-x_1(t)),$

$\dot{x}_2(t)=x_2(t)(G-k),$

$.\dot{x}_3(t)=-\alpha x_3(t)+x_2(t)(Gu(t)-1+e^{-0.1375t})$

In the above model, G is a demand function. Since in the economics world, the supply and demand model is set for the competitive market where neither the buyers nor the sellers can have a drastic effect on the price, and the price is regarded as a data. The producer's produce and the consumer's demand are dependent on the market price of the product. The supply law states that if the other conditions are constant, the supply value is dependent on the price, and the higher the price; the more the supply and vice versa. The demand law also states that if other conditions are constant, at higher prices, demand is lower and at lower prices demand is higher. In the competitive market, the equilibrium price and the equilibrium amount are determined by the supply anddemand for that good in the market. At higher prices, there is a shortage of demand and a surplus of supply. This overhead puts pressure on prices and pushes the price back to equilibrium. At lower prices, the demand is more than the supply causing a surplus of demand. This demand surplus will increase the price and therefore returns the price to its former value (equilibrium price). Once the price has reached a balance, this price will last forever.
In general, in the insurance market, the negative law of the demand also applies. Premiums assigned to the demand law are proportional to the amount of interest, i.e., people are more willing to purchase an insurance policy with lower premium. One of the solutions to this problem is to consider G's function in terms of other variables such as premium, the market average premium, or claim size rate which we will focus onin the next section.

SOLVING THE MODEL

The model explained in Section 3, solving via PMP theorem and following algorithm and the results described in the next steps will be achieved. According to PMP, we consider the following equations:

$$\dot{p}_i=-\frac{\partial H}{\partial x_i}\ , \tag{7}$$

$$\frac{\partial H}{\partial u^*}=0 \tag{8}$$

- H is 'Hamiltonian function'.
- u(t) is 'Control variable'.
- x_i are states of the system.
- We consider p_i as the 'Lagrange multipliers' and 'co-states' of the system.

The algorithm used to solve the model is as follows:

1. Subdivide the interval $[t_0, t_f]$ into N equal subintervals and assume a piecewise-constant control $u^{(0)}(t)=u^{(0)}(t_k)$, $t\in[t_k,t_{k+1}]$ $k=0,1,\ldots,N-1$.
2. Applying the assumed control u^i to integrate the state equations from t_0 to t_f with initial conditions $x(t_0)=x_0$ and store the state trajectory $x^{(i)}$.
3. Applying u^i and x^i to integrate costate equations backward, i.e., from $[t_0,t_f]$. The initial value $p^i(t_f)$ can be obtained by:

$$p^i(t_f)=\frac{\partial h}{\partial x}(x^{(i)}(t_f)). \tag{9}$$

Evaluate $\partial H^i(t)/\partial u$, $t\in[t_0,t_f]$ and store the vector.

4. If

$$\left\|\frac{\partial H^i}{\partial u}\right\| \leq \gamma \tag{10}$$

$$\left\|\frac{\partial H^i}{\partial u}\right\|^2 = \int_{t_0}^{t_f}\left[\left\|\frac{\partial H^i}{\partial u}\right\|\right]^T\left[\left\|\frac{\partial H^i}{\partial u}\right\|\right] dt \tag{11}$$

Then stop the iterative procedure. Here γ is a preselected small positive constant used as a tolerance. If (10) is not satisfied, adjust the piecewise-constant control function by:

$$u^{i+1}(t_k)=u^i(t_k)-\tau\frac{\partial H^i}{\partial u}(t_k), \quad k=0,1,\ldots,N-1 \tag{12}$$

Replace $u^{(i)}$ by $u^{(i+1)}$ and return to step2. Here, τ is the step size. In order to solve the problem and implement the above algorithm using sample data set in table 2, we verify in the two following cases:

- Case I: We set the initial conditions, designed state and demand function as $x(0)=(0.099,0.7,0.9)$, $x_d=0.99$ and $G=\frac{1}{e^{u(t)}}$ respectively. Then the problem is solved and the value of the objective function is obtained $J = .0429$. The state and control function are showed in Fig 1 and 2.
- Case II: We set the initial conditions, designed state and demand function as $x(0)=(0,0.75,0.77)$, $x_d=0.9$ and $G(t)=1-u(t)$ respectively. Then the problem is solved and the value of the objective function is $J = 0.0377$. The state and control function are showed in Fig 3 and 4.

CONCLUSION

In this paper, an optimal control problem was proposed based on an insurance model. Althoughthe deterministic control theory has been used to identify an optimal premium strategy for an insurer to minimise wealth creation, in reality, the process of receiving premiums and paying losses have been described via a dynamic system. We therefore solve the model to determine insurance premiums using a sample dataset.

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Table 1: the loss rate

year	Month	Losses car insurance	year	month	Losses car insuance
2012	April	6640	2014	August	3933
	May	6911		September	4493
	June	7106		October	4154
	July	7503		November	3860
2013	April	4148		December	4187
	May	4865		January	4367
	June	4442		February	4173
	July	4843		March	3866
	August	4452	2015	April	3113
	September	4439		May	3437
	October	4190		June	3323
	November	3560		July	3278
	December	3531		August	3655
	January	3444		September	3620
	February	3580		October	3414
	March	3284		November	3410
2014	April	3151		December	3882
	May	3393		January	3877
	June	3431		February	3963
	July	3737		March	3963

Table 2: sample data set

Constant	value
Time horizon T	1 year
Depreciation of wealth α	0.05 p/a
Demand parameterisation a	1 p/a
Demand parameterisation b	1 p/a
Length of policy $l = k^{-1}$	1 year
Rate of market reaction λ	0.1 p/a
Loss ratio γ	0.9 p/a

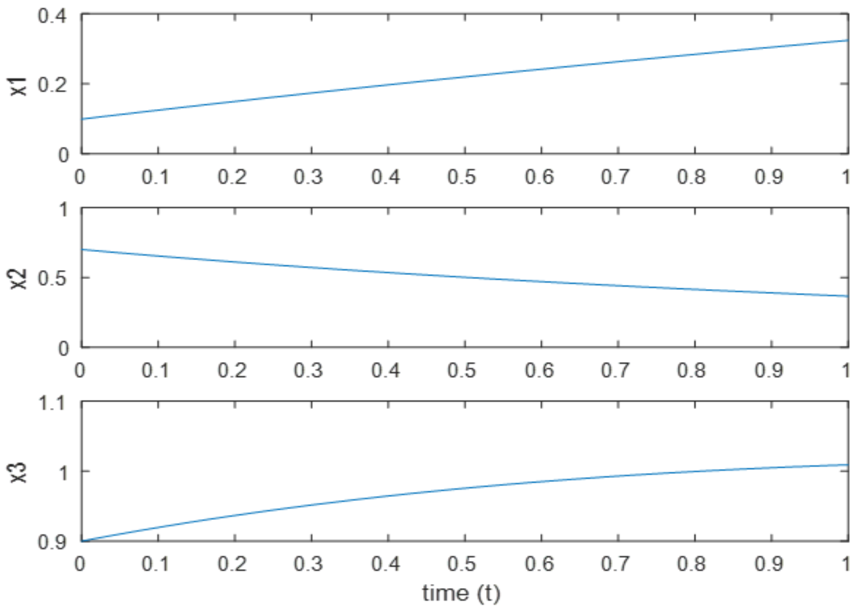

Figure 1: state equation

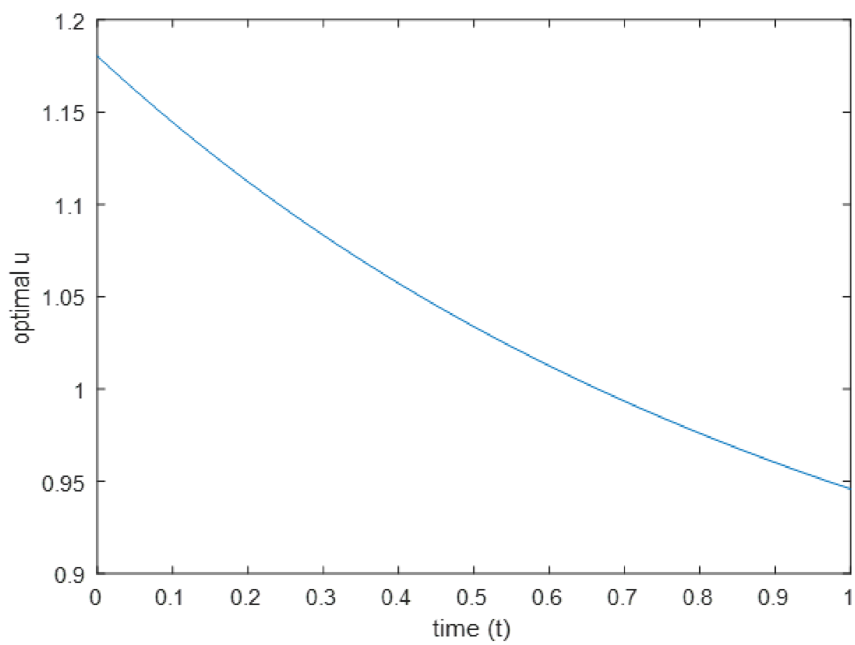

Figure 2: control function

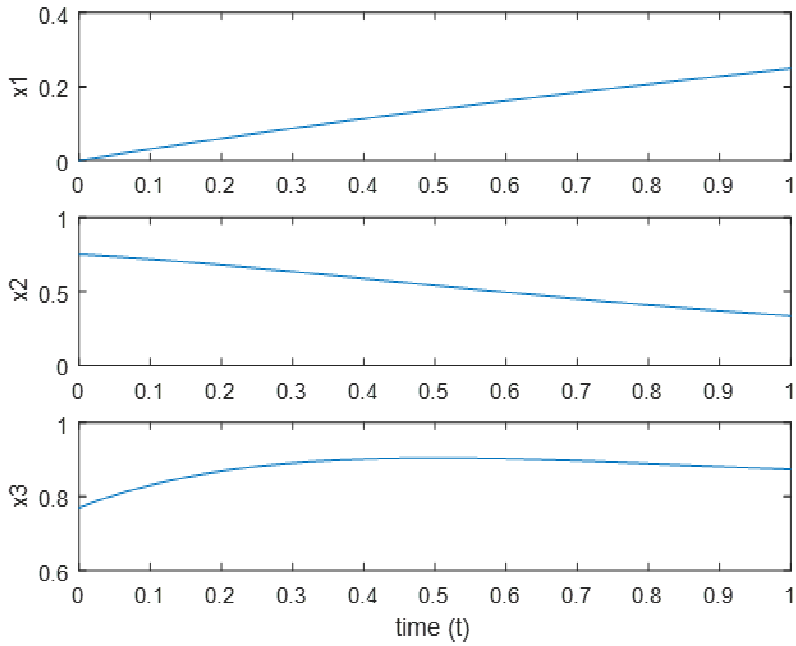

Figure 3: state equation

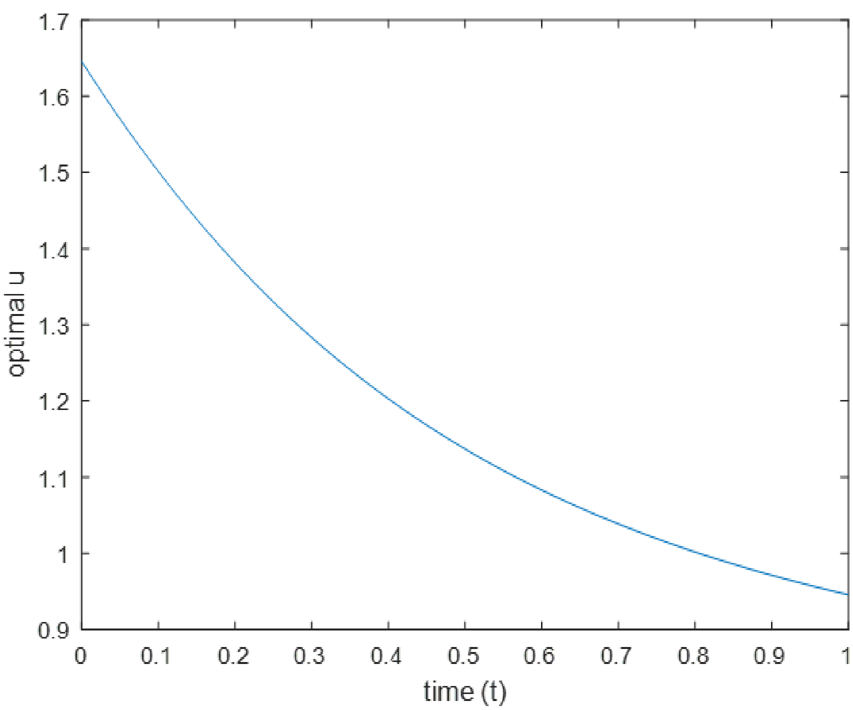

Figure 4: control function