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FRAME IN HILBERT SPACE AND ITS PROPERTIES

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ABSTRACT

*Since the 1950s, frames have been introduced as a good replacement for the feet and have been used as important and useful tools in signal processing, image processing, and so on. Interesting and profound results have been achieved in recent years by introducing operator theory and * c-algebra into frame studies . Continuous frames feature such properties as saturation, exhibitive sizes as well as confusion in certain states. In other words, these concepts are defined and their characteristics are discussed herein for Hilbert spaces that might be inseparable and indexed by a sizeable space like (Ω, μ). Some of the properties are similar to a discrete state but some of the others may have a greater deal of complexity. A frame makes it possible for each member of the display space to obtain members according to that frame. This is possible using dual frame definition, but it is often difficult or even impossible to obtain a dual frame. Accordingly, we will introduce frames with the behavior and features of the dual frame approach.*

Keywords: *Hilbert space, Frame operator, Continuous frame, Discrete frame, Frame properties.*

INTRODUCTION

A discrete frame is a continuation of $\{x_n\}$ from amongst the vectors in Hilbert Space, H, with such a characteristic as the existent of constants A and B in such a way that $A\|x\|^2 \leq \sum_j \left|\langle x, x_J \rangle\right|^2 \leq B\|x\|^2$ for all of the *x*s in Hilbert space.

Continuous frames can be defined based on a similar method.

The discontinuous and continuous frames not only play an important role in pure mathematics but they are also used in signals and images' processing and information compaction of the sampling theory and wavelet analysis. A reference that has useful information in theoretical regards to the frames' theory is the article titled "art of the frame theory" by Casazza (Benedetto and Fickus, 2003).

From applied perspectives, as well, another article by the same author under the title of "a novel instrument for the frame theory of Will-Heisenberg" can be pointed out.

Every discrete frame on Hilbert space is a combination of Riesz bases for a larger Hilbert space and each binary pair of the frames is the binary pair of the Riesz bases.

The present article deals with the frames in Hilbert space so as to prove the connection between them through gaining insight over the general properties of Hilbert Space, Bessel sequences and Riesz bases.

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Discrete Frame and its Frame Operator:

Definition One: family $\{f_{\ i}\}_{i\in I} \subseteq H$ is called a frame for Hilbert Space, H. If A and B be positive values in such a way that $\forall f\epsilon H: A\|f\|^2 \leq \sum_{i\in J} \ |<f, f_i>|^2 \leq B\|f\|^2$, they are termed the frame's boundaries: A is the lower boundary and B is the upper boundary (Casazza, 2001).

Definition Two: in defining a frame, if A=B, then $\{f_i\}_{i=1}^{\infty}$ is called a tight frame; if A=B=1, it is termed normalized tight frame or Parseval frame (Casazza, 2001).

Definition Three: the family $\{f_{\ i}\}_{i\in I} \subseteq H$ is called an exact frame if the existent elements do not form a frame with the elimination of any of the sequence elements (Casazza, 2001).

General Theorem One: the frames are complete in Hilbert space.

Definition Four: if $\{f_i\}_{i=1}^{\infty}$ be a frame for Hilbert Space, H, the frame operator is defined in the following form:

$$s_f = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i$$

Theorem Two: if $\{f_i\}_{i=1}^{\infty}$ be a sequence on Hilbert Space, H, the following terms will be subsequently holding true:

A) A frame in Hilbert Space, H, will be the one with boundaries A and B.

B) $S_f = \sum \ <f, f_i> f_i$ is a bounded linear operator with such a property as $AI \leq S \leq BI$.

Definition Five: $\{f_{\ i}\}_{i\in N} \subseteq H$ is called Bessel sequence when B>0 is existent as a constant in such a way that $\sum_{n\in N} |\langle f, f_n \rangle|^2 \leq B\|f\|^2$. (Casazza, 2001)

Theorem Three: if S be the operator of the frame $\{f_i\}_{i=1}^{\infty}$, then:

A) S can be inversed and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

B) $\{S^{-1}f_i\}_{i\in I}$ is a frame with the boundaries of A^{-1} and B^{-1}.

C) Every $f \in H$ can be written in the form of $f = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i \rangle S^{-1} f_i$ which is called frame decomposition operator.

Theorem Four: assume $\{f_i\}_{i=1}^{\infty} = \{T_i\}_{i=1}^{\infty}$ is a Riesz basis for H and T is an $\{e_i\}_{i=1}^{\infty}$ operator and an orthogonal basis. In this case, A and B constants would be existent in such a way that $\forall f\epsilon H: A\|f\|^2 \leq \sum_{i\in J} \ |<f, f_i>|^2 \leq B\|f\|^2$.

Bessel Sequences and Riesz Bases:

Definition Six: $\{f_i\}_{i=1}^{\infty}$ sequence is called a Riesz basis in Hilbert Space, H, if the orthonormal basis $\{f_i\}_{i=1}^{\infty}$ be existent along with the invertible and bounded operator in such a way that $T: H \to H$ for every$T_{e_i} = f_i; i \in N$.

Riesz Basis and Their Relationships with Frames:

Theorem Five: assume $\{f_i\}_{i=1}^{\infty}$ is a frame for Hilbert space, H, with A and B as boundaries; then, the following conditions hold:

A) $\{f_i\}_{i=1}^{\infty}$is a Riesz basis for Hilbert Space, H.

B) $\{f_i\}_{i=1}^{\infty}$ is an exact frame.

(Ω, μ)-Frame and Its Operator:

Here, we deal with the expressing of the theorems and the more general states of the frames, i.e. (Ω, μ)-Frame and its operator and properties (Casazza and Kutyniok, 2004).

Definition Seven: assume H is a sizeable Hilbert space (Ω, μ). Graph $F: \Omega \rightarrow H$ is called a (Ω, μ)-Bessel. If the constant A>0 be; then, $\int_{\Omega} |\langle x, F(\omega)\rangle|^2 \, d\mu\,(\omega) \leq A\|x\|^2$ for every $x\epsilon H$.

Definition Eight: assume H is a Hilbert Space and (Ω, μ) is a sizeable space. Graph $F: \Omega \rightarrow H$ is called (Ω, μ)-frame if constants B>0 and A are existent in such a way that:

$$\forall \omega \in \Omega B\|x\|^2 \leq \int |\langle x, f(\omega)\rangle|^2 \, d\mu\,(w) \leq A\|x\|^2 \text{ for every } x\epsilon H.$$

A (Ω, μ)-frame is called a tight frame F if A=B and it is called normalized tight frame if A=B=1. According to the definitions, if F be a (Ω, μ)-Bessel, the $T_F: H \rightarrow L^2(\Omega, \mu)$ operator is defined as $T_F x(\omega) = (\langle x, F(\omega)\rangle \omega \in \Omega)$ which is well-defined, linear and bounded.

$T_f^* T_f$ is a positive invertible operator on H and F is a normalized tight frame if and only T_F frame conversion be isometric, i.e. $|\langle T_F x, T_F x\rangle| = \left\|T_f\right\|^2 = \|x\|^2$ for every $x.y \in H$.

Therefore, Frame F is a normalized tight frame for H if and only if $\langle x, y\rangle = \langle T_F x, T_F y\rangle = \int \langle x, F(w)\rangle \langle F(w), y\rangle \, d\mu(\omega)$ for every $x.y \in H$.

Definition Nine: the two F and G (Ω, μ)-frames are similarly termed for Hilbert Spaces H and K if there is an invertible $S: H \rightarrow K^2$ operator in such a way that SF(ω) $= G(\omega), \omega \in \Omega$.

Theorem Six: the two (Ω, μ)-frames are similar on Hilbert Space if and only if their frame conversions feature similar ranges of values.

Definition Ten: if there is only one binary for (Ω, μ)-frame, it can be accordingly stated that the F is a frame of Riesz type.

Theorem Seven: assume F is a (Ω, μ)-frame in which case the following conditions hold:

A) F is a Riesz-type (Ω, μ)-frame.

B) T_F operator is surjective.

Theorem Eight: the binary G is a standard one for (Ω, μ)-frame F if and only if we have $||T_G x|| \leq ||T_D x||$ for each D binary of F.

Theorem Nine: assume $\{e_j\}$ is an orthonormal basis for H in which case the following conditions hold.

A) F is a normalized tight (Ω, μ)-frame for H.

B) There is an orthonormal set $\{\psi_i\}_{i\in J}$ in $L^2(\Omega,\mu)$ in such a way that we have $\sum_{j\in J}\left|\Psi_j(w)\right|^2 < \infty$ and

C) $F(\omega) = \sum_{j\in J} \psi_{\breve{j}(\omega)} e_j$ for almost all $\omega \in \Omega$.

Theorem Ten: the following conditions hold:

A) There is a Riesz-type (Ω, μ)-frame.

B) There is also an orthonormal basis $\{\psi_j\}$ in such a way that

C) $\sum_{j\in 5}\left|\Psi_j(\omega)\right| < \infty, w \in \Omega.$

D) $\{\psi_j\}_{j\in J}$ and $\sum_{j\in J}\left|\Psi_j(w)\right|^2 < \infty$ for every $\omega \in \Omega$.

Theorem Eleven: imagine F and G are (Ω, μ)-frames for H with $F(\omega) = \sum_{j\in J} \phi_j(\omega)e_j$ and $G(\omega) = \sum_{j\in J} \phi_j(\omega)f_j$ being respectively pertaining to the orthonormal bases $\{e_j\}$ and $\{f_j\}$ on H. So, G is a binary of F if and only if

$< \varphi_j, \psi_j > = < e_j, f_j >.$

In fact, if $e_j = f_j$ is selected, G is subsequently a binary of F if and only if $\{\varphi_j\}$ and $\{\psi_j\}$ are orthogonal.

Theorem Twelve: for every normalized tight (Ω, μ)-frame F on H space, we have $dim\, H = \int_\Omega \|F(w)\|^2\, d\mu\,(\omega)$.

Expansion:

In this section, we deal with the stating of the strongly disjoint and strongly complete theorems and their important and essential theoretical propositions are expressed and proved.

Definition Eleven: assume F and G are two (Ω, μ)-frames respectively for Hilbert Spaces H and K. F and G are called strongly disjoint if Range (T_g) is perpendicular to Range (T_f) and they are called disjoint if

$$Range(T_F) \cap Range\left(T_J\right) = \{0\}$$

and $Range(T_F) + Range(T_G)$ be closed subspaces of $L^2(\Omega,\mu)$; they are also called strongly complete if they are strongly disjoint and $F \oplus G$ be a of Riesz type (Ω, μ)-frame for $K \oplus H$.

Theorem Thirteen: assume F and G are (Ω, μ)-frames for H and assume also that A, B $\epsilon B(H)$ in such a way that $A^{*}A + B^{*}B = I$ is the difference in which case AF+BG is a normalized tight $(\Omega,$

μ)-frame for H. In fact, $\alpha F + \beta G$ is a normalized tight frame for H. so, we have $|\alpha|^2 + |\beta|^2 = 1$ for $\alpha, \beta \in B$.

Justification:

$$T_{AF+BG}: H \to L^2(\Omega, \mu)$$
$$\int |\langle x, (Af + BG)(\omega)\rangle|^2 \, d\mu(\omega) = \|x\|^2$$
$$|T_{AF+BG}(x)(\omega)| = |\langle x, (Af + BG)(\omega)\rangle|$$
$$\langle x, Af(\omega)\rangle\langle x, BG(\omega)\rangle$$
$$\big|T_{AF}(x)(\omega) + |T_{BG}(x)(\omega)|\big|$$
$$T_{AF+BG} = T_{AF} + T_{BG}$$

In this case, we have:

$$T_{AF}(x)(\omega) = T_F A^* x = \langle x, AF(\omega)\rangle = \langle A^* x, F(w)\rangle = (T_F A^* x)(\omega)$$

That results in:

$$\|T_F A^* x\|^2 + \|T_G B^* x\|^2$$
$$\alpha\tilde{\alpha} + \beta\tilde{\beta} = 1$$
$$A{=}\alpha I \to \tilde{\alpha} I$$
$$\beta = I \to B^* = \beta^* I$$
$$A^* A + \beta^* = \tilde{\alpha}\alpha I + \tilde{\beta}\beta I = I$$

Three-Theorem Fourteen: assume F and G are normalized tight (Ω, μ)-frames for H. In this case, F+G is a (Ω, μ)-frame for H.
Justification:
Assume that T_F and T_G exist. In this case, F=$T_F^* e_n$ and G=$T_G^* e_n$ wherein $\{e_n\}$ is an orthonormal basis.
In this case:

$$\int |\langle x, F + G\rangle|^2 \, d\mu(\omega)$$
$$\int |\langle x, T_F^* e_n > + < T_G^* e_n\rangle|^2 \, d\mu(\omega)$$
$$\int |\langle T_F(x), e_n\rangle|^2 \, d\mu(\omega) + \int |\langle T_G(x), e_n\rangle|^2 \, d\mu(\omega)$$
$$\int |\langle F(x)\rangle|^2 \, d\mu(\omega) + \int |\langle G\,(x)\rangle|^2 \, d\mu(\omega)$$
$$\|F\|^2 + \|G\|^2$$

Four-Theorem Fifteen: assume F is the (Ω, μ)-frame for Hilbert Space H and P is an image orthogonally projected from H onto the subspace M. In this case, pF and $p^{\perp}F$ are strongly disjoint if and only if $P(T_f^*T_f) = (T_f^*T_f)P$.

Justification:

pF and $p^{\perp}F$ are strongly disjoint if and only if $T_{p^F}x \perp T_{p^{\perp}}y$ for all of the $x.y \in H$.

$$< T_{p^F}x \perp T_{p^{\perp}}y >= \int_{\Omega} \langle x, pF(\omega)\rangle\langle p^{\perp} F(\omega), y\rangle \, d\mu(\omega)$$

$$\int_{\Omega} \langle px, F(\omega)\rangle\langle F(\omega), p^{\perp} y\rangle \, d\mu(\omega)$$

$$\langle T_f px, T_f p^{\perp} y\rangle$$

$$p^{\perp} T_f^* T_f \langle x, y\rangle p$$

In this case,

$$T_{p^F}x \perp T_{p^{\perp}}y$$

for $x.y \in H$ if and only if ……

In this case, $P(T_f^*T_f) = (T_f^*T_f)P$.

Seven-Theorem Sixteen: if F is a (Ω, μ)-frame for Hilbert Space H in such a way that $\dim(T_F H)^{\perp} < \infty$, there is an F₁ (Ω, μ)-frame for Hilbert Space K in such a way that $F \oplus F_1$ is a Riesz-type (Ω, μ)-frame.

Justification:

Assume $\{\psi_i\}$ is an orthonormal basis for Range $(T_F)^{\perp}$. In this case, F is a normalized tight (Ω, μ)-frame that has undergone frame conversion.

In this case, G=F$\oplus$ F_1 is a (Ω, μ)-frame for Hilbert Space K.

$Range(T_F)^{\perp} \oplus K = H$ in such a way that $Range(T_G) = L^2(\Omega, \mu)$. Since

$R(T_F) + M = R(T_G)$

$L^2(\Omega, \mu) = R(T_F) \oplus R(T_F)^{\perp} = R(T_G)$,

G is a Riesz-type (Ω, μ)-frame.

Theorem Seventeen: assume F is a (Ω, μ)-frame for H. The size of v for F can be displayed as an extreme point in $\{F, \mu\}$ if and only if

$Span\{< x, F(\omega) >< F(\omega), y >: x, y \in H\}$

be dense in $L^2(\Omega, \mu)$.

Justification: assume that v is an extreme point of $\{F, \mu\}$ and $Span\{< x, F(\omega) >< F(\omega), y > : x, y \in H\}$ is not dense in $L^2(\Omega, \mu)$. In this case, the function $g \in L^\infty(\Omega, \mu)$ would be obtained using Hanh-Banach Theorem in such a way that $g \neq 0$ and
$\int_\Omega \langle x, F(\omega)\rangle\langle F(\omega)\, , y\rangle g(\omega)\ d\mu\,(\omega) = 0$.
For all of the $x, y \in H$ and considering the conjugate of -g, it is assumed that conjugate of g is a real value and that $\|g\|_\infty \leq \frac{1}{2}$ in which case $v_1 = (1+g)v$ and $v_2 = (1-g)v$ are the projections of two different sizes for F although $V = 1/2(v_1) + 1/2(v_2)$ which is in conflict with the extreme point V.

$Span\{< x, F(\omega) >< F(\omega), y >: x, y \in H\}$ has to be dense in $L^2(\Omega, \mu)$.

On the other hand, it is assumed that $Span\{< x, F(\omega) >< F(\omega), y >: x, y \in H\}$ is dense in $L^2(\Omega, \mu)$ and $V = 1/2(v_1) + 1/2(v_2)$ projects the sizes of *v1* and *v2* that are absolutely convergent with it. In this case, there would be a function $g_1, g_2 \in L^\infty(\Omega, \mu)$ in such a way that $v_1 = g_1 v$ *{i=1,2}*. Since v_1 is a size projector for F, for all of the $x, y \in H$, it displays that

$$\int_\Omega \langle x, F(\omega)\rangle\langle F(\omega)\, , y\rangle\, d\mu\,(\omega)$$

$$\int_\Omega \langle x, F(\omega)\rangle\langle F(\omega)\, , y\rangle\, d\mu$$

$$\int_\Omega \langle x, F(\omega)\rangle\langle F(\omega)\, , y\rangle g_1\, d\mu\,(\omega)$$

Using $Span\{< x, F(\omega), y >: x, y \in H\}$ in $L^2(\Omega, \mu)$, we have $g_1 = 1$; then $v = v_1 = v_2$. Also v is an extreme point.
Lemma: assume that U is a linear operator on a Banach space X and assume that there is $\lambda_1, \lambda_2 \in \{0,1\}$ in such a way that $\left|\left|Ux - x\right|\right| \leq \lambda_1 \left|\left|x\right|\right| + \lambda_2 ||Ux||$ for all of the $x \in X$. In this case U is linear and invertible.

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \leq \|U_x\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|x\|$$

$$\frac{1-\lambda_2}{1+\lambda_1}\|x\| \leq \|U_x^{-1}\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|x\|$$

For all of the $x \in X$.

Theorem Eighteen: assume that F is a (Ω, μ)-frame on Hilbert Space H with A and B boundaries and also that $G: \Omega \to H$ is a function with a real value in such a way that $\lambda_1, \lambda_2 \geq 0$. So,

$max\left(\lambda_1 + \frac{d}{\sqrt{A}}, \lambda_2\right) < 1$

And

$$\int_{\Omega} f(\omega)(F(\omega) - G(\omega)d\mu(\omega) \leq \lambda_1 \left\| \int_{\Omega} f(\omega)F(\omega)d\mu(\omega) \right\| + \lambda_2 | \int_{\Omega} f(\omega)G(\omega)d\mu(\omega) \ + d\|f\|$$

for all of the $f \in L^2(\Omega, \mu)$ with $\mu(supp(f))$. In this case, G is a (Ω, μ)-frame for H with frame boundaries

$$A\left(1 - \frac{\lambda_1+\lambda_2+\frac{d}{\sqrt{A}}}{1+\lambda_2}\right), B\left(1 - \frac{\lambda_1+\lambda_2+\frac{d}{\sqrt{B}}}{1-\lambda_2}\right).$$

Justification:

Assume G is a Bessel-sequence. Using the assumptions $f \in L^2(\Omega, \mu)$ and $\mu(supp(f))$ and we also have:

$$\left\| \int_{\Omega} f(\omega)G(\omega)d\mu(\omega) \right\|$$

$$\left\| \int_{\Omega} f(\omega)F(\omega)d\mu(\omega) \right\| + \left\| \int_{\Omega} f(\omega)F(\omega) - G(\omega)d\mu(\omega) \right\|$$

$$\leq 1 + \lambda_1 \left\| \int_{\Omega} f(\omega)F(\omega)d\mu(\omega) \right\| + \lambda_2 | \int_{\Omega} f(\omega)G(\omega)d\mu(\omega) \ + d\|f\|$$

That results in:

$$\left\| \int_{\Omega} f(\omega)G(\omega)d\mu(\omega) \right\| \leq \frac{1+\lambda_1}{1-\lambda_2} \left\| \int_{\Omega} f(\omega)F(\omega)d\mu(\omega) \right\| + \frac{d}{1-\lambda_2}\|f\|$$

$$\leq \frac{1+\lambda_1}{1-\lambda_2}\|fT_F^*\| + \frac{d}{1-\lambda_2}\|f\|$$

$$\frac{1+\lambda_1}{1-\lambda_2}\sqrt{B} + d1 - \lambda_2\|f\|$$

Where, T_f is the operator of the frame under F.

Assume that $X = \{f \in L^2(\Omega, \mu), \mu\left(supp(f)\right) < \infty\}$.

$U: X \rightarrow H$ is defined by means of:

$$U(f) = \int_{\Omega} f(\omega)F(\omega)d\mu(\omega)$$

$$\|U\| = \frac{1+\lambda_1}{1-\lambda_2}\sqrt{B} + \frac{d}{1-\lambda_2}$$

Since X is dense in $L^2(\Omega, \mu)$, it can be concluded that U is uniquely converted from a bounder operator in $L^2(\Omega, \mu)$ to a soft operator on Space H.
Now, for every $x \in H$ and every $f \in X$, we have:

$$\langle f, U_x^* \rangle = \langle Uf, x \rangle$$

$$< \int_{\Omega} f(\omega)G(\omega)d\mu(\omega), x >$$

$$< \int_{\Omega} f(\omega)G(\omega) > d\mu(\omega)$$

That results in

$$U^*x =< G(\omega), x >$$

In this case:

$$\int_{\Omega} |< x, G(\omega) >|^2 \, d\mu(\omega)$$

$$\|U^*\|^2 \leq \|U^*\|\|x\|^2$$

$$\leq B\left(1 - \frac{\lambda_1 + \lambda_2 + \frac{d}{\sqrt{B}}}{1-\lambda_2}\|x\|^2\right)$$

Hence, B becomes the Bessel-sequence of the upper boundary.
It can be shown that G possesses a lower boundary. Assume that $S = (T_F^* T_F)^{-1}$ and $v = UT_f \, s$.
In this case, $SF(\omega)$ is the focal binary of F and also:

$$x = \int_{\Omega} \langle x, SF(\omega)F(\omega) \rangle d\mu(\omega)$$

$$vx{=}UT_f S_x = \int_{\Omega} \langle x, SF(\omega)F(\omega) \rangle d\mu(\omega)$$

for all of the $x \in H$.

In this case, according to Theorem Fifteen,

$$\|x - vx\| \le \lambda_1\|x\| + \lambda_2\|vx\| + d\left\|T_f S_x\right\| \le \left(\lambda_1 + \frac{d}{\sqrt{A}}\right)\|x\| + \lambda_2\|vx\|$$

is invertible for all of the $x \in H$. And,

$$\|x\| \le \frac{\lambda_1+1+\frac{d}{\sqrt{A}}}{1-\lambda_2} \text{ and } \|v\|^{-1} \le \frac{1+\lambda_2}{-\lambda_1+1+\frac{d}{\sqrt{A}}}.$$

In this case, $x = vv^{-1}x = UT_f Sv^{-1}x = \int_\Omega d\mu(\omega) < v^{-1}x, SF(\omega) > G(\omega)$.
We have,

$$\|x\|^4 = |\langle x, x\rangle|^2$$

$$\left|\int_\Omega \langle v^{-1}x, SF(\omega) >< G(\omega), x\rangle \, d\mu\,(\omega)\right|^2$$

$$\le \left|\int_\Omega \langle v^{-1}x, SF(\omega)\rangle\right|^2 d\mu(\omega) \int_\Omega |< x, G(\omega) >|^2 \, d\mu(\omega)$$

$$\le \frac{1}{A}\|v^{-1}x\|^2 \int_\Omega |\langle x, G(\omega)\rangle|^2 \, d\mu\,(\omega)$$

$$\frac{1}{A}\left[\frac{1+\lambda_2}{1-\left(\lambda_1+\frac{d}{A}\right)}\right]^2 \|x\|^2 \int_\Omega |< x, G(\omega) >|^2 \, d\mu(\omega)$$

$$\int_\Omega |< x, G(\omega) >|^2 \, d\mu(\omega) \ge A(1 - \frac{\lambda_1+\lambda_2+\frac{d}{\sqrt{A}}}{1+\lambda_2})^\wedge 2\|x\|^2$$

Conclusion One: assume F and G are in theorem thirteen. In this case, $U = T^*G$ and $v = T_G^* T_F (T_F^* T_F)^{-1}$ hence it can be concluded that $V = T_F^* T_F = T_G^* T_F$ in which case $T_G^* T_F$ is invertible.

Conclusion Two: assume F and G in Conclusion One with d=0. In this case F and G are similar.

Justification: assume that *Range* (T_F) $f \perp$ so$T_F^* f = 0$. In this case, it can be concluded that $\left|\left|T_G^* f\right|\right| \le \lambda_2 \left|\left|T_G^*\right|\right|$ so $T_G^* f = 0$ since $\lambda_2 < 1$ in which case $R(T_G) \subseteq R(T_F)$. Using the results of the Theorem Fifteen, it can be concluded that there is an inverse operator in D in such a way that DG is a binary of F because$R(T_{DG}) = R(T_G)$. This shows that DG is a binary of F so $T_{DG}) \subseteq$

$R(T_F)$. Using the Theorem Fourteen, $R(T_{DG}) = R(T_F)$ in which case T_G and T_F have identical range spaces.

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