Örgütsel Davranış Araştırmaları Dergisi
Journal Of Organizational Behavior Research
Cilt / Vol.: 4, Sayı / Is.: S2, Yıl/Year: 2019, Kod/ID: 91S2547
2528-9705

OPTIMAL BUDGETING BY USING A GENERALIZED OPTIMAL SYSTEM DESIGN DEA MODEL AND THE EFFECT OF A NEW DMU'S BUDGET IN THE MODEL

Alireza FAKHARZADEH JAHROMI*, Mohammad Hassan MOJTABAEI, Mina HAGHGUYAN JAHROMI

Department of OR, Faculty of Mathematics, Shiraz University of Technology, Shiraz, Iran, Fars Elites Foundation, P.O. Box 71966-98893, Shiraz, Iran.

***Corresponding Author**
Email: Afakharzadeh@sutech.ac.ir

ABSTRACT

As there are two constraints in the design of an optimal system via data envelopment analysis (OSDviaDEA) models, a maximum of two responses are determined for prioritizing optimal budgeting allocations. While for the organization administrators, determining the budget in an optimal way for each individual making unit (DMU) is considered an important issue. Likewise, determining the optimal budget of the entirety of an organization, the wasted budget and /or budget deficit are decisively important issues to the managers. Thus, due to the fact that the OSD DEA models can merely decide on two optimal conditions for the organizational DMUs, they do not have the necessary efficiency in this regard. The present research paper sets out to introduce an extended OSD DEA to resolve some of the shortcomings of the former models. Moreover, the effects of adding a new DMU in prioritizing optimal budgeting of the units through extended OSD DEA are explored. And in continuation, drawing upon the sensitivity analysis procedure on LP problem, a novel method is presented which helps to prevent solving the problem for a second time. The final part is dedicated to providing numerical examples for explaining the proposed method.

Keywords: *Optimal budgeting; Generalized OSDDEA method; linear programming; Data envelopment analysis.*

INTRODUCTION

One of the impediments encountered by administrators in various industries is that of designing optimal systems, especially when the accessible budget is an established one. However, effective and easily implementable procedures are not readily attainable to overcome this obstacle. So, this article is an attempt to obtain an extension model based on data envelopment analysis (DEA) for optimal system design (OSD). DEA is a nonparametric method applied in such diverse fields as economics, industrial engineering and operations research whose goal is to estimate production frontiers. It is utilized empirically to measure the productive efficiency of DMUs. Despite the fact that DEA has a strong link to production theories in economics, the approach could be used for benchmarking in operations management, where a set of measures is selected to benchmark the performance of manufacturing and service operations. Within DEA, predefined systems (DMUs) are taken into account through LP models so as to make a distinction between optimizing a predefined system and designing an optimal one (Zelený, 1990). After all, designing optimal systems is a problem

Geliş tarihi/Recieved: 10.01.2019 – Kabul tarihi/Accepted: 03.05.2019 – Yayın tarihi/Published: 21.05.2019

common to many fields of endeavor. A large amount of literature is devoted to accomplishing the OSD through experimentation and/ or mathematical modeling which is applied in such structured systems as cooling towers, data storage modules, etc (Alavi, Rahmati and Ziaei-Rad, 2017; Mortazavi, Karbasian and Goli, 2016; Rahmati, Alavi and Ziaei-Rad, 2017; Agarwal, 2016; Abdel-Aleem et al., 2017).

Indeed, DEA is an applied procedure which can be used in diverse branches. As a quick review of some of the recent research works using DEA approach, one can refer to Lee and Pai (2015) who applied improved DEA and VIKOR for assessing the performances of TFT-LCD producers in Taiwan and Japan from 2002 to 2009; Pan et al. (2010) employed the DEA procedure using common DEA models to measure the performance distinction value between Asian and European countries; Yang (2013) presented a procedure to investigate the results of an environmental control utilized by a production unit to measure the opportunity cost of environmental regulations (OCER); Azadi and Saen (2012) proposed a slacks-based measure (SBM)-undesirable output model to help administrators ascertain the best suppliers in the presence of stochastic data, and undesirable parameters as well. The corresponding deterministic data as nonlinear programming factors are also derived; Shirouyehzad et al. (2014) presented a novel method for investigating the sensitivity of inefficient models. The main advantage of such a method is the flexibility obtained in the process of decision making. They further applied the model in the hotel industry to demonstrate its capability. Aksezer and Benneyan (2010) evaluated the entirety of the hospital performance under a unit owner: They applied DEA via missing data methods. White and Bordoloi (2014) provided a review and classification of the literature in relation to applying DEA to a variety of resource and budget allocation managements to improve efficiency. Oh and Shin (2015) applied a Monto Carlo analysis to present measurement errors: They used DEA as a budgeting strategy.

Furthermore, Wei and Chang (2011) developed effective approaches providing help to decision makers in coping with the challenges of designing their optimal systems and determining the corresponding optimal budgets. They utilized DEA, de novo programming and parametric linear programming through a parametric right-hand side, in order to create a design of an optimal system through data envelopment analysis (OSD DEA) models.

Considering the research studies conducted within OSD DEA models, it should be noted that-due to the accessibility of two constraints only-the optimal budget allocation can be established for a maximum of two DMUs. Besides, one cannot find comprehensive model in the literature for the determination of the optimal budget should new DMUs arise. In OSD DEA models, when a new DMU is added to the organization, all calculation processes should be executed all over again so as to determine total optimal budget, the which procedure brings about a waste of time and an increase in expenditures. So, it should be of utmost importance to administrators to know the optimal budget for all DMUs on a separate basis, in addition to determining the optimal budget of the whole organization, the wasted budget, and the budget deficit. But the presented OSD DEA model in (Wei and Chang, 2011) does not work efficiently in these regards, for it can determine the optimum conditions for two DMUs at most. And so, the present paper aims to improve and resolve the deficiencies present in OSD DEA models set forth in (Wei and Chang, 2011). Our research is organized as follows: After a short introductions provided at the beginning of section 1, section 2 is devoted to explaining OSD

DEA model in brief. Section3 introduces an algorithm for our proposed extended OSD DEA-hence forth GOSD DEA model. To continue, section 4 is an attempt to present a solution to the problem of allocating optimal budget based on data envelopment analysis in case a new DMU emerges; in such a way that the optimal budget appropriation to other DMUs is not changed. Numerical examples are also furnished in this section. Conclusions remarks in section 5 bring this research study to an end.

Optimal system design DEA model in brief

This section is intended to re redefine the OSD DEA model briefly. The proof of theorems and lemma are omitted from the discussion. As a matter of fact, in order to construct OSD DEA model, the researchers have utilized DEA, de novo programming, and parametric linear programming with a parametric right-hand side. In order to introduce the latter model, it is essential to define, the production possibility set. If m inputs and s outputs are assumed to be available under the postulates envelopment, convexity, output inefficiency, and minimum extrapolation, a generalized production possibility set T is indicated as follows:

$$T = \{(x,y)| \sum_{j=1}^{n} \hat{x}_j \lambda_j = x , \sum_{j=1}^{n} \hat{y}_j \lambda_j \ge y , \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge 0, j = 1,2, \dots , n\}. \tag{1}$$

Where $\hat{x}_j = (\hat{x}_{1j}, \hat{x}_{2j}, \dots , \hat{x}_{mj})^T > 0$ and $\hat{y}_j = (\hat{y}_{1j}, \hat{y}_{2j}, \dots , \hat{y}_{sj})^T > 0$ are respectively the input and output vectors for the jth DMU. It is also assumed that the target DMU's price data are accessible. An $m \times 1$ vector $C = (c_1, c_2, \dots , c_m)^T > 0$ and an $s \times 1$ vector $P = (p_1, p_2, \dots , \dots , p_s)^T > 0$ are assumed to represent the input and output prices, respectively. It should be noted that the reference DMUs price data are not known to the target DMU. Consequently, if the DMU's total accessible budget β is known, an OSD DEA model can be established to maximize the DMU's revenue as in the following:

$$\begin{aligned} \text{Max } & P^T y \\ S.to: & (x,y) \in T; \\ & C^T x \le \beta. \end{aligned} \tag{2}$$

Let n reference DMUs be considered existent; then, the target of the final OSD DEA model is to determine the optimal budget of DMUs, which is proposed in (Wei and Chang, 2011) according to the following model:

$$\begin{aligned} \text{Max } & (P^T \hat{Y})\lambda \\ S.to: & (C^T \hat{X})\lambda \le \beta; \\ & e^T \lambda \le 1; \\ & \lambda \ge 0 . \end{aligned} \tag{3}$$

If the objective function value in (3) is denoted by Z, in order to find out how the value of β affects the maximum revenue, β is considered as a variable and $Z = f(\beta)$; indeed, on the domain $\Omega = \{\beta | \beta \ge 0, \beta \in \mathbb{R}\}$, $f(\beta)$ is being an indirect production function. Therefore, Theorem 1 can evidently be proved.

Theorem 1. On the domain Ω, $f(\beta)$ is a monotonically non-decreasing concave function of β; moreover, $f(0) = 0$.

Definition1. If $f(\beta)$ denotes the optimal objective function in (3) and supposing that $\beta^{**} = \min\left\{\beta^* \middle| \max_{\beta \geq 0} f(\beta) = f(\beta^*)\right\}$, then β^{**} is defined as the optimal budget.

The following theorems and lemmas are provided in (Wei and Chang, 2011) to derive the optimal budget β^{**}.

Theorem 2. Assume that $\max_{1 \leq j \leq n} P^T \hat{y}_j = P^T \hat{y}_k, \beta^* = C^T \hat{x}_k$ and consider the following problem:

$$\begin{aligned} f(\beta^*) = \max\ &(P^T \hat{Y})\lambda \\ S.to{:}\ &(C^T \hat{X})\lambda \leq \beta^*; \\ &e^T \lambda \leq 1; \\ &\lambda \geq 0\,. \end{aligned} \tag{4}$$

Then,

(i) $f(\beta) = f(\beta^*)\ for\ any\ \beta \geq \beta^*$;

(ii) $f(\beta) \leq f(\beta^*)\ for\ any\ 0 \leq \beta \leq \beta^*$.

Lemma 1. If $\beta \geq \beta^*$, then the set of optimal solutions of (3) is $\{\sum_{k \in K} \alpha_k e_k | \sum_{k \in K} \alpha_k = 1,\ \alpha_k \geq 0, k \in K\}$, in which $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \Re^n$ (all elements are equal to zero except kth element that is equal to one) and $K = \left\{k \middle| \max_{1 \leq j \leq n} P^T \hat{y}_j = P^T \hat{y}_k\right\}$.

Theorem 3. If $K = \left\{k \middle| \max_{1 \leq j \leq n} P^T \hat{y}_j = P^T \hat{y}_k\right\}$ and $\min_{k \in K} C^T \hat{x}_k = C^T \hat{x}_k$, then $\beta^{**} = C^T \hat{x}_k$.

In the preceding part, the budget constraint $C^T x \leq \beta$ is defined by the suggested OSD DEA models. If condition $C^T x = \beta$ is considered, i.e. budget β has to be totally consummated and the following linear programming is obtained:

$$\begin{aligned} \hat{f}(\beta) = \max\ &P^T y \\ S.to{:}\ &\hat{X}\lambda = x\,; \\ &\hat{Y}\lambda \geq y; \\ &e^T \lambda \leq 1; \\ &C^T x = \beta; \\ &x \geq 0, \quad y \geq 0, \quad \lambda \geq 0. \end{aligned} \tag{5}$$

Suppose that there is a feasible solution for (5) with $\beta \geq 0$, $\bar{\lambda}$ being the optimal solution of (5) and a feasible scenario in which $\beta^* = C^T \hat{X} \bar{\lambda} < \beta$ is considered; then, $\hat{f}(\beta) \leq f(\beta)$ which means the optimal amount of (5) is lower than that obtained from the OSD DEA model. This phenomenon (not seeming to be natural) is actually deduced from the congestion, that is, when the condition $\beta > \beta^{\star} = \mathrm{C}^{\mathrm{T}}\widehat{\mathrm{X}}\bar{\lambda}$ holds, the congestion will definitely happen. This can be demonstrated in Theorem 4.

Theorem 4. Let $\bar{\lambda}$ be the optimal solution of (3) and $\beta^{\star} = C^T\hat{X}\bar{\lambda} < \beta$, then $\hat{f}(\beta) \le f(\beta) = f^{\star}(\beta)$.

The generalized optimal system design DEA (GOSDDEA) model

In order to design an extended version of OSD DEA model which can overcome the above draw backs, first, based on applying the theoretical points mentioned in the previous section, an algorithm for the OSD DEA procedure is introduced. Subsequently, a GOSD DEA model is presented to address the deficiencies.

- *The algorithm for implementing the OSD DEA method*

In light of the above-mentioned algorithms, based on the way the OSD DEA models are configure, the optimal budget might be presented through the following algorithm:

OSD DEA algorithm

Step1. (Preparation) Specifying the number of DMUs in the organization (*n*), presenting the total available budget for the organization (β), quantifying the inputs (*m*) and outputs (*s*) of the organization for each DMU separately and quantifying the values of inputs © and outputs (*P*) prices for each DMU.

Step2. (Modelling) Forming the OSD DEA model (3) and solving it.

Step3. (Optimal budgeting specified) From step 2, the nonzero optimal λ^{*}s and their respective DMUs are found; then, the desired amounts of the budget are specified against these DMUs. Now the maximum profit can be calculated.

It should be noted that practical examples are provided in (Wei and Chang, 2011) to illustrate how to implement the algorithm and how to determine the budgeting.

Introducing the GOSD DEA Method

As mentioned before by solving the OSD DEA model explained in the above section, at most two DMUs are determined for prioritizing optimal budget allocations, for model (3) merely has two constraints; it is of great importance for the administrator of an organization, however, to know the optimal budget for all DMUs separately. Besides, beyond determining the optimal budget of the whole organization, pinpointing the wasted budget and the budget deficit are equally crucial to an administrator. Hence, OSD DEA, as it stands, is not efficient and adequate enough to satisfy these requirements. In this regard, in this purls a GOSD DEA model is presented to overcome these drawbacks. Indeed the GOSD DEA method can specify the optimal budget for each DMU; if the available budget of the organization is lower than the minimum optimal budget of all DMUs, the organization will encounter a budget deficit. Also, if the optimal budget is allocated to all DMUs and there is a budget surplus, the organization will run across extra budget. In either case, after implementing the GOSD DEA method, the budget deficit or the budget surplus is exactly computable by subtracting the sum of the allocated optimal budgets from the available total budget. The GOSD DEA is presentable in the following manner.

Step1. Performing the OSD DEA algorithm introduced in section 3.1 (steps 1 to 3) to determine the optimal budget for at most two DMUs;

Step2. Specifying the new budget of the organization by subtracting the optimal budget of the optimally identified DMUs in the process of prioritizing the budget (obtained from the previous step) from the total available budget;
Step3. Eliminating the identified DMUs from the whole DMUs in the organization;
Step4. If the budget allotted to all DMUs is determined or the accessible budget is less than the minimum budget required for all the remaining DMUs, the algorithm will stop; otherwise, forming a new OSDDEA model by eliminating the related variable of the identified DMUs from the objective function and restrictions. Consider the new budget of the organization as the total available budget and return to step 1.

- ***Simulation and analysis***

In order to illustrate how to implement the GOSD DEA algorithm proposed for budgeting the organization, based on real data, a numerical example is investigated. The data are reported in Ertay et al. (2006) for a facility layout design (FLD) in a manufacturing plant. These data are collected from a SERT plastic industry company in SPAIN. FLD refers to coherent and harmonic establishment of machinery, equipment, sections, and workstations alongside of each other in a manufacturing DMU. This formation will result in the best efficiency realized in a combination of human resources, materials, equipment, and machinery for manufacturing products having maximum efficiency and profitability taking into account a set of goals, constraints, and some other conditions.

Table 1 contains the real data for 19 DMUs in an organization (19 FLDs) each having material handling cost and adjacency scores as input variables, and shape ratio, utilization of material-handling vehicle, flexibility and quality as output variables. If the total available budget for the organization (β) is assumed equivalent to the total sum of two types of inputs for all DMUs of FLD organization, (3) can be formed for prioritizing optimal budget designation.

It is assumed that the inputs and outputs do not outrank one another; accordingly, β =489639.26, C= (1, 1), and P= (1, 1, 1, 1) are the given values. By forming (3), and solving it using LINGO11 software in the first run, the 16th DMU is identified as the most efficient DMU in the organization (λ_{16}^{*}). Given this condition, when the optimal budget is allocated to the most efficient DMU, the optimal revenue will be $f(\beta) = 34.0821$. Now theorem 4 is employed to determine the optimal budget as:

$$K = \{16\} \quad , \qquad Min\, C^T \hat{x}_{16} = 26333.5 \quad , \quad \beta^{**} = 26333.5.$$

This is all that OSD DEA can achieve. That is, it cannot go any further whatsoever towards other DMUs performing additional analysis. Now let's consider what happens when GOSD DEA is used. By invoking GOSD DEA algorithm-via LINGO-and executing the optimal budgeting process for all DMUs, a set of totally different results are obtained. These data are provided in Table 2. As is inferred from the Table, the amount of the wasted budget is equal to $\beta - \sum_{i=1}^{n} \beta_i^{**} = 9952.24$, where β_i^{**} is the optimal budget of the *i*th DMU resulting from Theorem 4 forming the fourth column of the Table. Since this value is positive, the organization can have the same production while, consuming less budget. In other words, the proposed optimal budget for the organization resulting in a maximum revenue is equal to $\sum_{i=1}^{n} \beta_i^{**} = 479687.02$.

Comparing the obtained values to those of Ertay et al. (2006), it can be deduced that the identified DMU for prioritizing the budget matches the most efficient DMU identified in (Ertay et al., 2006) i.e., both methods introduced the DMU number 16 as the most efficient one-a predictable outcome.

Table 1. Reference (Ertay et al., 2006) organization data

DMUs	DEA inputs		DEA ouputs			
	Cost ($)	Adjacency score	Shape ratio	Flexibility	Quality	Hand-carry utility
1	20309.56	6405	0.4697	0.0113	0.041	30.89
2	20411.22	5393	0.438	0.0337	0.0484	31.34
3	20280.28	5294	0.4392	0.0308	0.0653	30.26
4	20053.2	4450	0.3776	0.0245	0.0638	28.03
5	19998.75	4370	0.3526	0.0856	0.0484	25.43
6	20193.68	4393	0.3674	0.0717	0.0361	29.11
7	19779.73	2863	0.2854	0.0245	0.0846	25.29
8	19831	5473	0.4398	0.0113	0.0125	24.8
9	19608.43	5161	0.2868	0.0674	0.0724	24.45
10	20038.1	6078	0.6624	0.0856	0.0653	26.45
11	20330.68	4516	0.3437	0.0856	0.0638	29.46
12	20155.09	3702	0.3526	0.0856	0.0846	28.07
13	19641.86	5726	0.269	0.0337	0.0361	24.58
14	20575.67	4639	0.3441	0.0856	0.0638	32.2
15	20687.5	5646	0.4326	0.0337	0.0452	33.21
16	20779.75	5507	0.3312	0.0856	0.0653	33.6
17	19853.38	3912	0.2847	0.245	0.0638	31.29
18	19853.38	5974	0.4398	0.0337	0.0179	25.12
19	20355	7402.01	0.4421	0.0856	0.0217	30.02

However, it should be noted that the novel method can additionally identify the wasted budget. In this way, the decision makers might investigate the congestion phenomenon in budgeting. So, after determining the amount of $\sum_{i=1}^{n} \beta_i^{**} = 479687.02$, the sensitivity analysis (Bazaraa, Jarvis and Sherali, 2011) can be performed for the previous problem. Now if the amount of available β is less than $\sum \beta^{**}$, the organization will encounter budget deficit. To take an example, if the predicted budget β is assumed to be 200000, under such condition, prioritizing budget for the organization DMUs is calculated after performing GOSD DEA algorithm as is given in the following:

$$\lambda_{16}^* = 1, \lambda_{15}^* = 1\,, \lambda_{14}^* = 1, \lambda_{17}^* = 1, \lambda_2^* = 1\,, \lambda_1^* = 1, \lambda_3^* = 1, \lambda_{19}^* = 0.7298985$$

Table 2. Results of prioritizing budget allocation for FLD organization DMUs

Available budget	λ^*	$f(\beta)$	β^{**}	$\beta - \beta^{**}$	DMUs
489639.26	$\lambda_{16}^* = 1$	34.0821	26333.5	463305.8	1
463305.8	$\lambda_{15}^* = 1$	33.7215	26333.5	436972.3	2
436972.3	$\lambda_{14}^* = 1$	32.6935	25214.67	411757.6	3
411757.6	$\lambda_{17}^* = 1$	31.8835	23765.38	387992.2	4
387992.2	$\lambda_{2}^* = 1$	31.8601	25804.22	362188	5
362188	$\lambda_{1}^* = 1$	31.412	26714.56	335473.4	6
335473.4	$\lambda_{3}^* = 1$	30.7953	25574.28	309899.2	7
309899.2	$\lambda_{19}^* = 1$	30.5694	27757.01	282142.1	8
282142.1	$\lambda_{11}^* = 1$	29.9531	24846.68	257295.5	9
257295.5	$\lambda_{6}^* = 1$	29.5852	24586.68	232708.8	10
232708.8	$\lambda_{12}^* = 1$	28.5928	23857.09	208851.7	11
208851.7	$\lambda_{4}^* = 1$	28.4959	24503.2	184348.5	12
184348.5	$\lambda_{10}^* = 1$	27.2633	26116.1	158232.4	13
158232.4	$\lambda_{5}^* = 1$	25.9166	24368.75	133863.6	14
133863.6	$\lambda_{7}^* = 1$	25.6845	22642.73	111220.9	15
111220.9	$\lambda_{18}^* = 1$	25.6114	25827.38	85393.53	16
85393.53	$\lambda_{8}^* = 1$	25.2636	25304	60089.53	17
60089.53	$\lambda_{9}^* = 1$	24.9188	25367.86	34721.67	18
34721.67	$\lambda_{13}^* = 1$	24.8766	24769.43	9952.24	19
9952.24					

Since $\beta < \sum \beta^{**}$, the last DMU (the 19th DMU in the organization) cannot gain its required budget in such prioritizing process. As a matter of fact, the available budget for the 19th DMU is equal to the designated amount 0.7298985 which is far less than that required for the latter DMU. In such a case, the organization faces a budget deficit. If the available β is higher than $\sum \beta^{**}$, all the DMUs will earn their optimal budget and an amount equivalent to $\beta - \sum \beta^{**}$ of the budget is wasted. When this happens, the designated budget exceeds the requirements of the organization.
In this application, the use of the OSDDEA model and the general algorithm (GOSDDEA) for allocating the optimal budget to all DMUs was introduced. After solving the problem and allocating the optimal budget, if a new DMU is added to the problem, using the sensitivity analysis can be obtained a limit in which the optimization conditions do not change the problem. In the following, the impact of adding a new DMU to the problem is presented.

Effect of adding a new DMU on designating the optimal budget

There is need for most organizations to establish new DMU(s) after sometime; the new DMU(s) require(s) budget allocation exactly like the previous ones. Accordingly, the optimal budget allotted to other DMUs will undergo some changes and a re-reallocated is required, the result of which is a waste of time and money; in this section, we are to obtain a solution to optimal budget designation drawing upon data envelopment analysis in case a new DMU is added. Charnes and Cooper were the first researchers who conducted an investigation on sensitivity analysis within DEA framework in 1968 (Charnes and Cooper, 1968). Then, the necessity for novel algorithms was discussed, for the data in both sides of the constrain of related linear programming problems in DEA are changed (Charnes et al., 1984). Next, Charnes and Neralic (Charnes and Neralic, 1992) developed a technique for carrying out a sensitivity analysis where simultaneous changes in all inputs and outputs of a DMU occur. After that, sensitivity and stability analyses in classifying the efficiencies related to data envelopment analysis models were discussed (Jahanshahloo et al., 2005). In what follows, we try to illustrate that without solving several linear programming problems, the effect of adding a new DMU on the preceding optimal budgeting could be investigated by introducing a budget area for the DMUs. If the added DMU belongs to this budget proper, then the optimal budget allotted to the previous DMUs, will remain unchanged; otherwise, we have to face changes in the budget allocations. It should be reminded that by specifying the use of sensitivity analysis in linear programming theory, this budget proper can be specified via a simple computing process in a short time. The process is accomplished using GOSD DEA, as a result of which, there is no need to solve the problem for a second time; thus, executing a high amount of computations is prevented. This investigation is inspired by the sensitivity analysis method in the LP theory and introduces a novel procedure having the stated characteristics.

- ***Modeling and investigating the effect of adding a new DMU***

Let us assume that there are n DMUs with $\widehat{\mathcal{X}_j} = (\widehat{\mathcal{X}}_{2j}, \widehat{\mathcal{X}}_{2j}, \dots \widehat{\mathcal{X}}_{ij}, \dots \widehat{\mathcal{X}}_{mj})^T > 0$ and $\widehat{y_j} = (\hat{y}_{2j}, \hat{y}_{2j}, \dots \hat{y}_{rj}, \dots \hat{y}_{sj})^T > 0$ as the input and the output vectors for the j^{th} DMU, respectively. Also let $\widehat{P} = (p_1, p_2, \dots, p_s)^T > 0$ be the weight (cost) vector corresponding to the outputs, $C = (c_1, c_2, \dots, c_m)^T > 0$ be the weight (price) vector corresponding to the inputs and β be the total budget of the organization. It should be mentioned that in this section the notation $\widehat{P}$ is used as the output weight vector in order to make difference with the coefficients of the objective function *P*. According to the definition of production possibility set in section 2, the OSD DEA model performs as following in order to maximize the revenue of the objective DMU and afterwards allocate the corresponding optimal budget:

$$\text{Max} \sum_{j=1}^{n} \hat{P}^T \hat{y}_{rj} \lambda_j \qquad r = 1,2,\dots,s \tag{6}$$

$$S.to: \sum_{j=1}^{n} \hat{C}^T \hat{x}_{ij} \lambda_j \leq \beta\,;\; i = 1,2,\dots,m;$$

$$\sum_{j=1}^{n} e^T \lambda_j \leq 1, \qquad j = 1,2,\dots,n;$$

$$\lambda_j \geq 0 \;\;, \qquad i = 1,2,\dots,m.$$

It is clear that when a DMU is added to model (6), actually a variable is added to the initial problem. This is equivalent to adding a constraint to the dual problem; then the dual of (6) in standard form is stated as:

$$\text{Min} \sum_{i=1}^{m} \beta v_i + \sum_{r=1}^{s} u_r \tag{7}$$

$$S.to: \sum_{i=1}^{m} C^T x_{ij} v_i + \sum_{r=1}^{s} e^T u_r - \sum_{r=1}^{s} \hat{P}^T y_{rj} - \bar{s}_j = 0;$$

$$\sum_{i=1}^{m} C^T x_{i,n+1} v_i + \sum_{r=1}^{s} e^T u_r - \sum_{r=1}^{s} \hat{P}^T y_{r,n+1} - \bar{s}_{n+1} = 0;$$

$$v_i\,, u_r\,, \bar{s}_j, \bar{s}_{n+1} \geq 0, \qquad i = 1,2,\dots,m\,, \qquad j = 1,2,\dots,n,$$

where $\mathrm{u_r}(\mathrm{r} = 1,2,\dots,\mathrm{s})$ and $\mathrm{v_i}(\mathrm{i} = 1,2,\dots,\mathrm{n})$ are dual variables. If B is the optimal bases matrix related to (7), then the base and the new vector are as follows:

$$B_{new} = \begin{pmatrix} B & 0 \\ N & -1 \end{pmatrix}, \qquad b_{new} = \begin{pmatrix} b \\ \sum_{r=1}^{s} \hat{P}^T y_{r,n+1} \end{pmatrix}; \tag{8}$$

$$N = \left[C^T x_{j1,n+1}\,, \dots, C^T x_{jL,n+1}, 1,1,\dots,1,0,0,\dots,0\ \right]$$

where L is the number of variables available in the base and the zeros of matrix N corresponding to auxiliary variables of the problem in the initial bases. In this regard:

$$(B_{new})^{-1} = \begin{pmatrix} B^{-1} & 0 \\ B^{-1}N & -1 \end{pmatrix}. \tag{9}$$

Since in any linear programming problem, adding a new variable may affect the optimality and feasibility of the preceding optimal solution, the conditions of optimality and feasibility of problem (7) should be investigated. Thus it is inferred from (Bazaraa, Jarvis and Sherali, 2011) that (10) is established for all the basic variables:

$$Z_i - C_i = [C_B, 0] \begin{pmatrix} B^{-1} & 0 \\ B^{-1}N & -1 \end{pmatrix} \begin{pmatrix} a_i \\ C^T x_{i,n+1} \end{pmatrix} - C_i = C_B B^{-1} a_i - C_i. \tag{10}$$

It shows that adding a new restriction does not have an impact on the optimality of the problem; for it does not bring about any changes in the coefficients of the basic variables. Therefore, the possibility condition under which the efficiency of the DMU in question is unchanged-when prioritizing budget allocations-a point be discussed below.

Theorem 5. If the new DMU_{n+1} satisfies the following inequality, the efficiency of the DMU under discussion, will remain unchanged:

$$\sum_{i=1}^{m} C^T x_{i,n+1} v_i^* + \sum_{r=1}^{s} u_r^* - \sum_{r=1}^{s} \hat{P}^T y_{r,n+1} \geq 0. \tag{11}$$

where $u_r^*(\mathrm{r} = 1,2, \dots, \mathrm{s})$ and $v_i^*(\mathrm{i} = 1,2, \dots, \mathrm{n})$ are the optimal dual variables.
Proof: Since adding a constraint does not influence the optimality of problem (7) (based on Eq. (10)), it is sufficient to investigate the feasibility condition. Thus, the new values of the right hand side are calculated as:

$$(B^{-1}b)_{new} = \begin{pmatrix} B^{-1} & 0 \\ B^{-1}N & -1 \end{pmatrix} \begin{pmatrix} b \\ \sum_{r=1}^{s} \hat{P}^T y_{r,n+1} \end{pmatrix} = \begin{pmatrix} \bar{b} \\ N\bar{b} - \sum_{r=1}^{s} \hat{P}^T y_{r,n+1} \end{pmatrix}. \tag{12}$$

Accordingly, for the feasibility of the solution, (13) must hold:

$$N\bar{b} - \sum_{r=1}^{s} \hat{P}^T y_{r,n+1} \geq 0. \tag{13}$$

If $u_1, u_2, \dots, u_k, k \in \{1,2, \dots, s\}$ and $v_1, v_2, \dots, v_L, L \in \{1,2, \dots, m\}$ are considered to be the initial basic variables, then,

$$N = [C^T x_{1,n+1}, \dots, C^T x_{L,n+1}, 1,1, \dots,1,0,0, \dots,0] \quad ,\bar{b} = [v_i^*, u_r^*, \bar{s}_t^*] \ . \tag{14}$$

Hence, $N\bar{b} = \sum_{i=1}^{L} C^T x_{i,n+1} v_i^* + \sum_{r=1}^{k} u_r^*$. However, the values of $u_r(r = k+1, \dots, s)$ and $v_i(i = L+1, \dots, m)$ are zero because they are non-basic variables; therefore,

$$N\bar{b} = \sum\nolimits_{i=1}^{m} C^T x_{i,n+1} v_i^* + \sum\nolimits_{r=1}^{s} u_r^* . \tag{15}$$

Now from (13) we have:

$$\sum\nolimits_{i=1}^{m} C^T x_{i,n+1} v_i^* + \sum\nolimits_{r=1}^{s} u_r^* - \sum_{r=1}^{s} \hat{P}^T y_{r,n+1} \geq 0 \ , \tag{16}$$

which is the condition stated in the theorem. □

Thus, if condition (13) holds, then adding DMU_{n+1} has not effect on the optimal solution; *i.e.*, the feasibility condition holds for the linear programming problem. If the conditions of

Theorem 5 hold for each DMU by determining their budget intersection, the range of changes related to the input and output of the new DMU with regard to affecting the new solution could be obtained (for the conditions are stated in terms of the unknown variables x and y). Furthermore, an appropriate range might be determined for the new DMU in a way that the allocated budget to the previous DMUs remains unchanged. The process of budgeting for the organization DMUs is explained through an example in the following section. Following that, the appropriate range for adding a new DMU is obtained by applying Theorem 5.

- ***Numerical results***

In this section, based on the working model in subsection 4.1, some numerical results are given for the test problems raised in (Wei and Chang, 2011) and (Ertay, Ruan and Tuzkaya, 2006). To that end, it is supposed that a new DMU is added to the above-mentioned organizations. To perform the necessary computations, software Lingo 11 is used on a PC with Intel Core i5–7200U CPU and 12GB of RAM by double precision format.

Example 1: An organization has 4 decision making units. The values for the input and output for each DMU are shown in Fig. 1. The available budget for the organization is assumed to be 10 units.

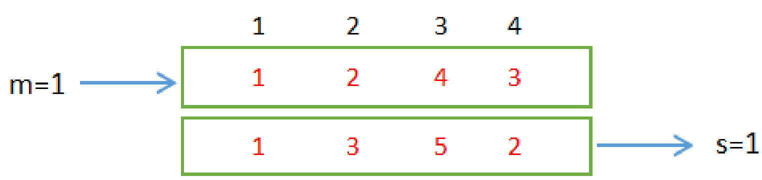

Figure 1: Input and output data of four decision making units in organization

New, we try to find the proper region for adding a DMU such that the allocated budget for the previous units remains unchanged. The linear programming problem (6) for this example runs as follows:

$$\begin{aligned} Max \quad & Z = \lambda_1 + 3\lambda_2 + 5\lambda_3 + 2\lambda_4 \\ S.to: \quad & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1; \\ & \lambda_1 + 2\lambda_2 + 4\lambda_3 + 3\lambda_4 \leq 10; \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0. \end{aligned} \tag{17}$$

By running the GOSD DEA algorithm, the results cited in Table 3 are obtained. These results are calculated by correcting the available budget and performing a repetitive process in optimal budget appropriation. Now, by applying Theorem 5 for each DMU we will have:

$$\begin{aligned} & DMU_3 \; : \; N\bar{b} - \textstyle\sum_{r=1}^{s} \hat{P}^T y_{r,n+1} = 5 - y \geq 0 \,; \textstyle\sum_{r=1}^{s} \hat{P}^T y_{r,n+1} = y \\ & DMU_2 \; : N\bar{b} - \textstyle\sum_{r=1}^{s} \hat{P}^T y_{r,n+1} = 3 - y \geq 0 \,; \\ & DMU_4 \; : N\bar{b} - \textstyle\sum_{r=1}^{s} \hat{P}^T y_{r,n+1} = 2 - y \geq 0 \,; \\ & DMU_1 \; : \; N\bar{b} - \textstyle\sum_{r=1}^{s} \hat{P}^T y_{r,n+1} = x - y \geq 0, \; N\bar{b} - \textstyle\sum_{r=1}^{s} \hat{P}^T y_{r,n+1} = 1 - y \geq 0 \,. \end{aligned} \tag{18}$$

Table 3. Results obtained by prioritizing optimal budget designation for organization

The available budget	λ^*	Z	β^{**}	$\beta-\beta^{**}$	step
10	$\lambda_3^*=1$	5	4	6	1
6	$\lambda_2^*=1$	3	2	4	2
4	$\lambda_4^*=1$	2	3	1	3
1	$\lambda_1^*=1$	1	1	0	4

Intersecting all the above inequalities would produce the set S= $\{(x,y)|y\leq 1, x\geq y\}$; therefore, S is the budget area for which the optimal budgeting process will not change if in terms of x (input) and y (output), the coordinates of the new decision making unit are substituted in that; otherwise, the budget designation process has to be repeated. The output of the new DMU should be less than or equal to 1 and the input should be larger or equal to the output such that the new DMU can be placed in the budget area S.

Example 2: Consider the organization referred to in section 3.3 along with the data in Table 1. We intend to find the proper region for adding a new DMU such that the allocated budget for the other units is unchanged. The obtained results for budgeting through our new method are presented in Table 2 (using the GOSD DEA algorithm). Now, by applying Theorem 5 for each DMU, the following results ensue.

$$\begin{aligned}
&DMU_{16}\ :\ N\bar{b}-\textstyle\sum_{r=1}^{s}\hat{P}^{T}y_{r,n+1}=34.0821-y\geq 0\,;\\
&DMU_{15}\ :\ N\bar{b}-\textstyle\sum_{r=1}^{s}\hat{P}^{T}y_{r,n+1}=33.7215-y\geq 0\,;\\
&\dots\\
&DMU_{13}\ :\ N\bar{b}-\textstyle\sum_{r=1}^{s}\hat{P}^{T}y_{r,n+1}=24.8766-y\geq 0\,;\\
&DMU_{1}\ :\ N\bar{b}-\textstyle\sum_{r=1}^{s}\hat{P}^{T}y_{r,n+1}=x-y\geq 0,\ N\bar{b}-\textstyle\sum_{r=1}^{s}\hat{P}^{T}y_{r,n+1}=1-y\geq 0\,;\\
&\text{where } \textstyle\sum_{r=1}^{s}\hat{P}^{T}y_{r,n+1}=y.
\end{aligned} \tag{19}$$

Intersecting all the above inequalities implies the set $=\{(x,y)|y\leq 24.8766\,,\ x\geq y\}$. Hence, the previous optimal budgeting process will not change if coordinates (x,y) in terms of x (input) and y (output) of the new decision making unit belong to S; otherwise, the budget appropriation process has to be repeated. In this regard, the output of the new DMU should be less than or equal to 24.8766 and its input should be larger or equal to the output such that the new DMU can be placed in the budget area S.

CONCLUSIONS

The percent article advances a new GOSD DEA procedure to address some of the limitations of the conventional OSD via DEA models which are only capable of handling two Decision Making Units (DMU). The introduced extended method can readily deal with prioritizing budget appropriations for other DMUs determining the optimal budget for each particular DMU. In addition, the proposed extended method is versatile; that is, it can meet several secondary goals includes optimal budgeting and maximum revenue of the whole organization when the optimal budget is determined. As each organization is designed to attain specific objectives in agreement its inherent characteristics, a sensitivity analysis is further performed to investigate the effect of adding a new DMU on the optimal budget allocated to the other divisions. As a result, a budget proper is obtained such that the optimal budget dedicated to other DMUs remain unchanged if coordinates of the new DMU are placed in the budget proper in terms of input and output; otherwise, budget has to be re-allocated. In general, a conclusion that can be drawn is that the presented method is capable of facilitating the budgeting process in a short while, thus empowering the organization's administrator with regard to decision making on budgetary problems by giving them broader perspectives and requisite insight.

References

Abdel-Aleem, A., El-Sharief, M.A., Hassan, M.A., El-Sebaie, M.G.: A surface response optimization model for EPQ system with imperfect production process under rework and shortage. Opsearch. (2017). doi: 10.1007/s12597-017-0301-1

Agarwal, S.: DEA-neural networks approach to assess the performance of public transport sector of India. Opsearch. 53(2), 248-258 (2016)

Aksezer, C.S., Benneyan, J.C.: Assessing the efficiency of hospitals operating under a unique owner: a DEA application in the presence of missing data. Int. J. Serv. Oper. Manag. 7, 53-75 (2010)

Alavi, S.R., Rahmati, M., Ziaei-Rad, S.: A new approach to design safe-supported HDD against random excitation by using optimization of rubbers spatial parameters. Microsyst. Technol. 23(6), 2023-2032 (2017)

Azadi, M., Saen, R.F.: Developing a new chance-constrained DEA model for supplier's selection in the presence of undesirable outputs. Int. J. Oper. Res. 13, 44-66 (2012)

Bazaraa, M.S., Jarvis, J.J., Sherali, H.D.: Linear programming and network flows. John Wiley & Sons (2011)

Charnes, A., Cooper, W.W., Lewin, A.Y., Morey, R.C., Rousseau, J.: Sensitivity and stability analysis in DEA. Ann. Oper. Res. 2(1), 139-156 (1984)

Charnes, A., Cooper, W.W.: Structural sensitivity analysis in linear programming and an exact product form left inverse. Nav. Res. Log. (NRL) 15(4), 517-522 (1968)

Charnes, A., Neralic, L.: Sensitivity analysis of the proportionate change of inputs (or outputs) in data envelopment analysis. Glas. Mate. 27, 393-405 (1992)

Ertay, T., Ruan, D., Tuzkaya, U.R.: Integrating data envelopment analysis and analytic hierarchy for the facility layout design in manufacturing systems. Inf. Sci. 176(3), 237-262 (2006)

Jahanshahloo, G.R., Lotfi, F.H., Shoja, N., Tohidi, G. and Razavyan, S.: Sensitivity of efficiency classifications in the inverse DEA models. Appl. Math. computation, 169(2), 905-916 (2005).

Lee, Z.Y. and Pai, C.C.: Applying Improved DEA & VIKOR Methods to Evaluate the Operation Performance for World's Major TFT–LCD Manufacturers. Asia-Pac. J. Oper. Res. (2015). doi: 10.1142/s0217595915500207

Mortazavi, S.M., Karbasian, M. and Goli, S.: Evaluating MTTF of 2-out-of-3 redundant systems with common cause failure and load share based on alpha factor and capacity flow models. Int. J. Syst. Assur. Eng. Manag. (2016). doi: 10.1007/s13198-016-0553-9

Oh, S.C., Shin, J.: The impact of mismeasurement in performance benchmarking: A Monte Carlo comparison of SFA and DEA with different multi-period budgeting strategies. Eur. J. Oper. Res. 240(2), 518-527 (2015)

Pan, T.W., Hung, S.W., Lu, W.M.: DEA performance measurement of the national innovation system in Asia and Europe. Asia-Pac. J. Oper. Res. 27(03), 369-392 (2010)

Rahmati, M., Alavi, S.R., Ziaei-Rad, S.: Improving the read/write performance of hard disk drives under external excitation sources based on multi-objective optimization. Microsyst. Technol. 23(8), 3331-3345 (2017)

Shirouyehzad, H., Hosseinzadeh Lotfi, F., Dabestani, R.: A modified model for sensitivity analysis of inefficient DMUs in DEA: a case study in hotel industry. Int. J. Oper. Res. 19(2), 186-197 (2014)

Wei, Q., Chang, T.S.: DEA-type models for designing optimal systems and determining optimal budgets. Math. Comp. Model. 54(11), 2645-2658 (2011)

White, S.W., Bordoloi, S.K.: A review of DEA–based resource and cost allocation models: implications for services. Int. J. Serv. Oper. Manag. 20(1), 86-101 (2014)

Yang, C.C.: A DEA-Based Approach for Evaluating the Opportunity cost of Environmental Regulations. Asia-Pac. J. Oper. Res. (2013). doi: 10.1142 / s0217595912 500492.

Zelený, M.: Optimizing given systems vs. designing optimal systems: The De Novo programming approach. Int. J. Gen. Syst. 17(4), 295-307 (1990)