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The use of homotopy Regularization Method for Linere and nonlinner Fredholm Integral Equations of the first kind

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ABSTRACT

Recently,Wazwaz has studied the regularization method to the one-dimensional linear Fredholm integral equations of the first kind in [5]. In this work, we develop this method to the linear and nonlinear two-dimensional Fred-holm integral equations of the first kind. Indeed, the regularization method is used for linear integral equations directly. But nonlinear integral equations of the first kind are transformed to linearintegral equations of the first kind by a change of variable, then The Regularization-Homotopy Method is applied.The combination of the regularization method and the homotopy perturbation method, or shortly, the regularization- homotopy method is used to find a solution to the equation. Some examples will be used to highlight the reliablity of the generalized The Regularization-Homotopy Method.

Keywords: *Fredholm integral equations; Regularization-homotopy method; Ill-posed problem.*

$$f(x,y) = \lambda \int_a^b \int_c^d k(x,y,s,t)dsdt \qquad (1)$$

1 Introduction

Integral equations of the first kind in one-dimensional case have been studied in many papers (see [8-14]). But although,this equations in two-dimensional case have many interesting applications in Mechanical engineering, Physical sciences and other applied sciences see [5,8], only a few papers have been written about them (see [1 - 13]).In this paper, we consider general form of the two-dimensional Fredholm integral equations of the first kind

where f and K are continuous functions and is a constant. Also,F is a continuous function which has continuous inverse and finally u is the unknown function of the equation (1) to be find. Obviously, if G is linear then the Eq. (1) wil be linear. As mentioned above, we develop the regularization method of [1] to linear case directly and in nonlinear case, we first set $u(x,t) = F(h(x,t))$ to convert (1) to linear form, then the

1.1 The homotopy perturbation method

The homotopy perturbation method was introduced and developed by Ji- Huan He in

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[14] and was used recently in the literature for solving linear and nonlinear problems. The homotopy perturbation method couples a homotopy technique of topology and a perturbation technique. A homotopy with an embedding parameterp[0,1] is constructed, and the impeding parameter p is considered a small parameter. The method was derived and illustrated in [14], and several differential equations were examined. The coupling of the perturbation method and the homotopy method has eliminated the limitations of the traditional perturbation technique [14]. In what follows we illustrate the homotopy perturbation method to handle Fredholm integral equations of the second kind and the first kind.

$$u_\alpha(x,y) = \frac{1}{\alpha} f(x,y) - \frac{\lambda}{\alpha}\int_a^b \int_c^d k(x,y,s,t)u_\alpha(s,t)dsdt \qquad (2)$$

obtained above in (2). The homotopy is now constructed

$$u_\alpha(x,y) = p\left(\frac{1}{\alpha} f(x,y) - \frac{\lambda}{\alpha}\int_a^b \int_c^d k(x,y,s,t)u_\alpha(,t)dsdt\right) \qquad (3)$$

where the embedding parameter p monotonically increases from 0 to 1. The homotopy perturbation method permits the use of the expansion

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_n^n\,(x,y) \qquad (4)$$

and consequently

$$u(x,y) = \lim_{p\to 1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_n^n\,(x,y) \qquad (5)$$

Substituting (4) into both sides of (3), and equating the terms with the same powers of the embedding parameter p the recurrence relation is obtained

$$p^0: u_0(x,y) = 0$$
$$p^1: u_1(x,y) = \frac{1}{\alpha}(x,y), \qquad (6)$$
$$p^{N+1}: u_{n+1}(x,y) = -\frac{1}{\alpha}\int_a^b \int_c^d k(x,y,s,t)dsdt,$$

Having determined the components$u_i(x,y), i \geq 0$, we then use

$$u(x,y) = \lim_{p\to 1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_n^n\,(x,y). \qquad (7)$$

The series (7) converges to the exact solution if such a solution exists. It is important to note that if the kernel is separable, i.e. $K(x,y,s,t) = g(x,y)h(s,t)$, then the following condition

$$\left|1 - \int_a^b \int_c^d k(x,y,s,t)dsdt\right| \qquad (8)$$

Note that the nonlinear Fredholm integral equations of the first kind can be handled by the regularization-homotopy method in a parallel manner to the analysis presented earlier for the linear case. The non-linear Fredholm integral equations of the first kind are first converted the non-linear Fred Holm integral equations of the first kind are first converted

$$f(x,y) = \int_a^b \int_c^d k(x,y,s,t)F(u(s,t)dsdt) \qquad (9)$$

to a linear Fredholm integral equations of the first kind of the form

$$f(x,y) = \int_a^b \int_c^d k(x,y,s,t)v(s,t)dsdt)\,, (x,y) \in D = [0,1][0,1] \qquad (10)$$

using the transformation

$$v(x,y) = F(x,y) \qquad (11)$$

Assuming that *F(u(x))* is invertible, then we can wirte

$$u(x,y) = F^{-1}\big(v(x,y)\big). \qquad (12)$$

The non-linear Fredholm integral equations of the first kind is often considered as an ill-posed problem and this may lead to several difficulties. In this work, we will limit ourselves only to cases where$K(x,t) = g(x)h(t)$. We will now examine the illustrative linear and non-linear Fredholm integral equations of the first kind.

2 LinearTwo-dimensional Fredholm integral equation of the first kind

Now, we are going to apply the regularization-homotopy method to illustrate the earlier presented analysis, for the linear case. Significantly, a necessary condition to guarantee a solution is that, the data function $f(x)$ must contain components which match the corresponding *x* components of the kernel $k(x,y,s,t) = g(x,y)h(s,t)$.

Example 2.1 *Use the* regularization-homotopy method to solve the linear Fredholm integral equations o fthe first ki

$$\frac{7}{12}(x+t) = \int_0^1 \int_0^1 (x+t)su(s,t)dsdt. \qquad (13)$$

Using the regularization method, Eq. (13) can be transformed to

$$u_\alpha(x,u) = p\left(\frac{7}{12\alpha}(x+t) - \frac{1}{\alpha}\int_0^1 \int_0^1 (x+t)su_\alpha(s,t)dsdt\right) \qquad (14)$$

We next construct the homotopy

$$u_\alpha(x,t) = p\left(\frac{7}{12\alpha}(x+t) - \frac{1}{\alpha}\int_0^1 \int_0^1 (x+y)su_\alpha(s,t)dsdt\right). \qquad (15)$$

Proceeding as before, we find the recurrence relation

$$\begin{aligned}
&p^0: u_0(x,y) = 0 \\
&p^1: u_1(x,y) = \frac{7}{12\alpha}(x+y), \\
&p^2: u_2(x,y) = -\frac{1}{\alpha}\int_0^1 \int_0^1 (x+y)su_1(s,t)dsdt = -\frac{49}{144\alpha^2}(x+y)\,. \\
&p^3: u_3(x,y) = -\frac{1}{\alpha}\int_0^1 \int_0^1 (x+y)su_2(s,t)dsdt = \frac{343}{1728\alpha^3}(x+y) \qquad (16) \\
&p^4: u_4(x,y) = -\frac{1}{\alpha}\int_0^1 \int_0^1 (x+y)su_3(s,t)dsdt = -\frac{2401}{20736\alpha^3}(x+y)
\end{aligned}$$

and so on. Based on this, we obtain the approximate solution

$$u_\alpha(x,u) = \frac{7}{12\alpha}(x+y)\left(1 - \frac{7}{12\alpha} + \frac{49}{144\alpha^2} - \frac{343}{1728\alpha^3} + \dots\right) \quad (17)$$

This in turn gives

$$u_\alpha(x,u) = \frac{7}{12\alpha+1}(x+y) \quad (18)$$

obtained upon summing the in nite geometric series. The exact solution u(x) of (13) can be obtained by

$$u_\alpha(x,u) = \lim_{\alpha\to 0} u_\alpha(x,u) = x+y. \quad (19)$$

Example 2.2 *Use the* regularization-homotopy method to solve the Linear Two-dimensional Fredholm integral equation of the first kind

$$\frac{1}{9}xy = \int_0^1\int_0^1 (xystu(s,t)dsdt. \quad (20)$$

by the using homotopy method the eq(20) converted to the second kind of equation

$$\alpha u_\alpha(x,y) = \frac{1}{9}xy - \int_0^1\int_0^1 (xyst)\, u_\alpha(s,t)dsdt, \quad (21)$$

So that

$$u_\alpha(x,u) = \frac{1}{9\alpha}xy - \frac{1}{\alpha}\int_0^1\int_0^1 (xyst)u_\alpha(s,t)dsdt, \quad (22)$$

Next, to construct the homotopy

$$u_\alpha(x,y) = p(\frac{1}{9\alpha}xy - \frac{1}{\alpha}\int_0^1\int_0^1 (xyst)u_\alpha(s,t)dsdt, \quad (23)$$

Proceeding as before, we find the recurrence relation

$$p^0: u_0(x,y) = 0$$
$$p^1: u_1(x,y) = \frac{1}{9\alpha}(x,y),$$
$$p^2: u_2(x,y) = -\frac{1}{\alpha}\int_0^1\int_0^1 xystu_1(s,t)dsdt = -\frac{1}{81\alpha^2}xy\,.$$
$$p^3: u_3(x,y) = -\frac{1}{\alpha}\int_0^1\int_0^1 xystu_2(s,t)dsdt = +\frac{1}{729\alpha^3}xy \quad (24)$$
$$p^4: u_4(x,y) = -\frac{1}{\alpha}\int_0^1\int_0^1 xystu_3(s,t)dsdt = -\frac{1}{7461\alpha^4}xy$$

And so on. Based on this, we obtain the approximate solation

$$u_\alpha(x,y) = \frac{1}{9\alpha}xy\left(1 - \frac{1}{9\alpha} + \frac{1}{81\alpha^2} - \frac{1}{729\alpha^3} + \dots\right), \quad (25)$$

This in turn gives

$$u_\alpha(x,y) = \frac{xy}{1+9\alpha} \quad (26)$$

obtained upon summing the infinite geometric series

$$u(x,y) = \lim_{\alpha\to 0} u_\alpha(x,y) = xy. \quad (27)$$

3 Nonlinear Tow-dimensional Fredholm integral equation of the first kind

The regularization-homotopy method is applied to illustrate the analysis presented before for the non-linear case, as given bellow. However, our focus will be limited to the separable kernel $k(xyst) = g(xy)h(st)$.

Example 3.1 Consider the following two-dimensional nonlinear Fredholm integral equation of the first kin

$$xy = 2\int_0^1\int_0^1 (xyst)u^4(x,t)dsdt \tag{28}$$

We first set

$$u(x,y) = \pm\sqrt[4]{u(x,y)} \tag{29}$$

to carry out the non-linear equation (28) to the linear Fredholm integral equation

$$xy = 2\int_0^1\int_0^1 (xyst)\, v(x,t)dsdt, \tag{30}$$

The regularization method carries Eq. (30) to

$$u_\alpha(x,y) = \frac{1}{\alpha}xy - \frac{2}{\alpha}\int_0^1\int_0^1 (xyst)\, v_\alpha(s,t)dsdt, \tag{31}$$

We next construct the homotopy

$$v_\alpha(x,y) = (\frac{1}{\alpha}xy - \frac{2}{\alpha}\int_0^1\int_0^1 (xyst)\, v_\alpha(s,t)dsdt), \tag{32}$$

Proceeding as before, we find the recurrence relation

$$\begin{aligned}
&p^0: v_0(x,y) = 0\\
&p^1: v_1(x,y) = \frac{1}{\alpha}xy\\
&p^2: v_2(x,y) = -\frac{2}{\alpha}\int_0^1\int_0^1 xystv_1(s,t)dsdt = -\frac{2}{9\alpha}xy\\
&p^3: v_3(x,y) = -\frac{1}{\alpha^2}\int_0^1\int_0^1 xystv_2(s,t)dsdt = +\frac{4}{81\alpha^3}xy\\
&p^4: v_4(x,y) = -\frac{2}{\alpha}\int_0^1\int_0^1 xystv_3(s,t)dsdt = -\frac{8}{729\alpha^4}xy.
\end{aligned} \tag{33}$$

and so on. Based on this, we obtain the approximate solution

$$v_\alpha(x,y) = \frac{1}{\alpha}xy\left(1 - \frac{2}{9\alpha} + \frac{4}{81\alpha^2} - \frac{8}{729\alpha^3} + \ldots\right) \tag{34}$$

This in turn gives

$$v_\alpha(x,y) = \frac{9xy}{9\alpha + 2} \tag{35}$$

obtained upon summing the infinite geometric series. The exact solution v(x) of (27) can be obtained by

$$v\ \ (x,y) = \lim_{\alpha\to 0} v_\alpha(x,y) = \frac{9}{2}xy \tag{36}$$

The exact solution u(x) of (28) can be obtained by

$$u(x,y) = \pm\sqrt[4]{\frac{9}{2}xy}. \quad (37)$$

Example 3.2 Consider the following nonlinear Fredholm integral equation of the first kind

$$\frac{x}{6(1+x)} = \int_0^1\int_0^1 \frac{x}{1+y}(1+s+t)u^2(s,t)dsdt. \quad (38)$$

We first transform the nonlinear Equation (38) to a linear equation by using the change of variable

$$v(s,t) = u^2(s,t) \quad (39)$$

So that Equation (38) becomes

$$\frac{x}{6(1+x)} = \int_0^1\int_0^1 \frac{x}{1+y}(1+s+t)v(s,t)dsdt. \quad (40)$$

We first set

$$u(s,t) = \pm\sqrt{v(s,t)} \quad (41)$$

The regularization method transform Equation (40) to

$$v_\alpha(x,y) = \frac{x}{6\alpha(1+x)} - \frac{1}{\alpha}\int_0^1\int_0^1 \frac{x}{1+y}(1+s+t)v_\alpha(s,t)dsdt. \quad (42)$$

Now, to construct homotopy we have

$$p^0: v_{\alpha,0}(x,y) = 0$$

$$p^1: v_{\alpha,1}(x,y) = \frac{x}{6\alpha(1+y)}$$

$$p^2: v_{\alpha,2}(x,y)$$

$$= -\frac{x}{6\alpha(1+y)}\int_0^1\int_0^1 (1+s+t)v_{\alpha,1}(s,t)dsdt$$

$$= -\frac{x}{6\alpha^2(1+y)}\left(\frac{3+2\,log(2)}{6}\right) \quad (44)$$

$$p^3: v_{\alpha,3}(x,y) = -\frac{x}{6\alpha(1+y)}\int_0^1\int_0^1 (1+s+t)v_{\alpha,2}(s,t)dsdt = \frac{x}{6\alpha^3(1+y)}\left(\frac{3+2\,log(2)}{6}\right)^2$$

$$p^4: v_4(x,y) = -\frac{x}{6\alpha(1+y)}\int_0^1\int_0^1 (1+s+t)v_{\alpha,3}(s,t)dsdt = \frac{x}{6\alpha^4(1+y)}\left(\frac{3+2\,log(2)}{6}\right)^3$$

Thus, the approximate solution becomes

$$v_\alpha(x,y) = \frac{x}{6\alpha(1+y)}\left(1 - \frac{1}{\alpha}\left(\frac{3+2\,log(2)}{6}\right)\frac{1}{\alpha^2}\left(\frac{3+2\,log(2)}{6}\right)^2 - \frac{1}{\alpha^3}\left(\frac{3+2\,log(2)}{6}\right)^3 + \cdots\right) \quad (45)$$

This in turn gives

$$v_\alpha(x,y) = \frac{x}{(1+y)(6\alpha + 3 + 2\, log(2))} \qquad (46)$$

Letting a ! 0, we obtain the exact solution as

$$v \ \ (x,y) = \lim_{\alpha\to 0} v_\alpha(x,y) = \frac{x}{(1+y)(3+2\, log(2))} \qquad (47)$$

Since

$$v(x,y) = \pm\sqrt{v(x,y)}. \qquad (48)$$

The exact solution u(x) of (38) can be obtained by

$$v \ \ (x,y) = \pm\sqrt{\frac{x}{(1+y)(3+2\, log(2))}} \qquad (49)$$

Conclusion

In this work, a combination of the regularization method and the homotopy perturbation method was proposed as a reliable treatment of the Two-dimensional linear and non-linear Fredholm integral equations of the first kind. The proposed method showed reliability to handling these ill-posed problems. three examples, linear and non-linear, were examined to illustrate the analyses which were presented. The exact solutions were formally derived, if the exact solutions existed, as these equations were ill-posed.We pointed out that the corresponding analytical solutions are obtained using Mathematica

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