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Numerical solution of linear and nonlinear fractional integrodifferential equations using Haar wavelets and Legendre substitution fraction functions

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ABSTRACT

We tackle the problem of nonlinear fractional integrodifferential equations with starting conditions using Legendre alternative functions in this study. To solve these equations, a new set of Legendre-alternative functions composed of Legendre-alternative polynomials is utilized. To solve these equations, a new set of Legendre-alternative functions composed of Legendre-alternative polynomials is utilized. The approach entails first constructing the Riemann-Liouville operational matrix, which has all of these fractional functions as components and then solving these equations using this matrix and the method of local points derived from the roots of Legendre polynomials. Then, the integral-differential fractions of Volta-Fredelm and Volta fractions are solved using the Haar wavelet. The Hare approach converts these equations into a system of linear algebraic equations that the Gaussian elimination method solves. The convergence of these two techniques is then investigated.

Keywords: *Fractional Integro-Differential Equations, Caputo Derivative, Operational Matrix, Wavelet, Legendre Alternative Functions.*

INTRODUCTION

Mathematics, as the mother of sciences, has influenced the development of many other disciplines. Basic sciences like physics, chemistry, and engineering sciences are the ones that owe the most to mathematics, particularly applied mathematics. Because fractional calculus explores in more depth the behaviors of functions that are the outcome of modeling natural processes, it has ushered in a revolution in computational applied sciences in recent decades. Fractional calculus is an extended version of ordinary calculus [1]. As a result, dealing with the numerical solution of fractional integrodifferential equations is critical since finding an analytical solution to these equations of physical origin is difficult or impossible. Fractions are used in a lot of the approaches for numerically solving integrodifferential equations. However, each numerical approach has its own set of disadvantages. Model equations, including partial differential equations, are transformed into discrete equation models, which appear as a collection of linear or nonlinear algebraic equations using numerical techniques. The computational cost of a big machine will grow if this set of algebraic equations is solved directly due to the considerable processing time and memory required for mathematical operations.
Several numerical findings on fractional-order derivatives have recently been gathered; for more information, check references [4-2]. The fundamental benefit of this approach (Haar method) is its simplicity and reduced computing costs, which are dependent on the dispersion of modified matrices and small significant numbers of wavelet coefficients, as compared to numerical

schemes in the fractional integrodifferential equation. Because boundary conditions necessitate precision, this approach makes it considerably easier to solve issues with limited values.

In addition, the authors of reference [5] developed a mathematical model for a tool known as a micro-electromechanical device. The gadget was created to determine the non-absorbency of fluids encountered during the oil extraction process.

It's worth noting that the development of fractional calculus has made it possible to describe the behavior of numerous devices that use fractional integral-differential and fractional differential devices. To overcome these challenges, a number of numerical approaches have been offered. For example, Katsikadyls [6] provided a technique for solving fractional differential equations based on deductive equations. The authors of reference [7] solved the integrodifferential equation using the human decomposition approach. Khader and Svilum employed the Chebyshev quasi-spectral approach to solving the system of linear and nonlinear fractional integrodifferential equations numerically in the reference [8].

We recommend the reader to reference [9] to consider the existence and uniqueness of the solutions of the integral-differential Voltaire-Fredelm Kasra equations. As a result, the purpose of this chapter is to define novel functions for solving nonlinear fractional integrodifferential equations based on alternate Legendre polynomials. To solve integral-differential fraction equations, this approach is accurate and simple to use. The integer-differential equations of linear and nonlinear fractions were numerically solved in this work using [10] and [11], using rabies and Legendre functions to substitute the fraction.

fractional-Integral differential equations

Many fields, including engineering, genetics, astronomy, electrochemistry, biology, and even economics, utilize fractional integral-differential equations. Many scientists from diverse domains have sought to solve these equations.

Definition 1: A fractional integrodifferential equation is one in which fractional derivatives of an unknown function are present in the integral equation. In general, these equations can be expressed as follows:

$$D^{\xi}y(x) = F(x, y, \int_{f(x)}^{g(x)} k(x,t)Y\big(t, y(t)\big)dt), \qquad \xi \in (n-1, n], n \in \mathbb{N}$$

Theorem 1: (Banakh theorem): If X is a full meter space and $f: X \to X$ is a contractional map, then f has a unique fixed point in X.

Newton's method for solving nonlinear devices $f(x) = 0$

Nonlinear device:

$$f(x) = \begin{bmatrix} f_1(x^1, \dots, x^n) \\ \vdots \\ f_n(x^1, \dots, x^n) \end{bmatrix} = 0$$

Suppose $x = \xi$ is a root of the device $f(x) = 0$. Newton's method for solving this device is as follows:

Suppose x_0 is an approximation of ξ. Create the sequence {x_n as follows:

$x_{n+1} = x_n - [Jf(x_n)]^{-1} f(x_n), \quad n = 0, 1, 2, \dots,$

Where

$$Jf(x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x^1}(x_n) & \cdots & \frac{\partial f_1}{\partial x^n}(x_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x^1}(x_n) & \cdots & \frac{\partial f_n}{\partial x^n}(x_n) \end{bmatrix}$$

Under appropriate conditions, the sequence {x_n will converge to the root of the nonlinear device f (x) = 0, i.e., ξ.

An efficient algorithm for the numerical solution of fractional integrodifferential equations by Wavelet

In this part of the Haar Wavelet method using co-local points to solve the integrodifferential equations of the Volta fraction in the form of:

$$\begin{cases} D^{\xi}y(x) = r(x) + b(x)y(x) + \int_0^x W(x,\tau)y(\tau)d\tau, x\epsilon[0,1], \\ y(0) = y_0, \end{cases} \quad (1)$$

And the integral-differential equation of the Volta-Fredelm fraction is as follows:

$$\begin{cases} D^{\xi}y(x) = r(x) + b(x)y(x) + \int_0^x W_1(x,\tau)y(\tau)d\tau + \int_0^1 W_2(x,\tau)y(\tau)d\tau, \\ y(0) = y_0, x\epsilon[0,1], \end{cases} \quad (2)$$

Which we use in reference [12],

Where y (x), 0 <ξ≤1 are unknown functions, W_2 and W_1 are the cores of the integral equations in relations (2) and (1), and b (x) and r (x) are known functions.

We make the procedure for ϵξ (0,1] and then continue in the same style for ξϵ (1,2], and if this method is possible, it can be expanded with fractional-order derivatives that have more initial conditions.

We use Banakh's theorem for the existence and uniqueness of the answer.

Theorem 2 - Suppose that (a) and (b) hold. If $\kappa = \frac{b^*+w_1^*+w_2^*}{\Gamma(\xi+1)} < 1$, then a unique answer y, Where yϵC([0,1] , R) exists for (2):

It is worth mentioning that:

C ([0,1], R) represents the set of all derivative functions from [0,1] to R.

Proof: Clearly, C ([0,1], R) is the space of Banach (because it is a perfect term vector space). An operator

$$T: C([0,1] , R) \rightarrow C([0,1] , R)$$

Is defined as follows:

$$Ty(x) = y_0 + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}r(s)ds + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}b(s)y(s)ds + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\left(\int_0^s W_1(s,\tau)y(\tau)d\tau + \int_0^1 W_2(s,\tau)y(\tau)d\tau\right)ds, \quad (8)$$

Where xϵ[0,1].

If y ϵ C([0,1] , R) a fixed point T, then y will be the answer (2). Suppose,

Suppose $y, \tilde{y} \in C([0,1], R)$:

Then:

$$\|Ty - T\tilde{y}\| = \sup_{x\in[0,1]} \left|\frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1} b(s)[y(s)-\tilde{y}(s)]ds \right. \quad (3)$$

$$+\frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\left(\int_0^s W_1(s,\tau)\quad [y(\tau)-\tilde{y}(\tau)]d\tau\right.$$

$$\left.\left. + \int_0^1 W_2(s,\tau)[y(\tau)-\tilde{y}(\tau)]d\tau\right)ds\right| \leq \left(\frac{b^* + W_1^* + W_2^*}{\Gamma(\xi+1)}\right)||y - \tilde{y}|| = \kappa\|y-\tilde{y}\|.$$

We know that according to the concepts of analysis, the function $||.||$ is equal to the distance between two points in multidimensional space.

Since the supremium of a set is the smallest boundary above that set, so $\sup_{x\in[0,1]} |.|$ ‹, The smallest distance of the set $\sup_{x\in[0,1]} |.|$, Which is the same as $\|Ty - T\tilde{y}\|$

Here $||.||$ is the symbol of the second term in Euclidean's next N vector.

As a result, T is a contraction map, which means that there is a unique answer of (2).

Haar Waves

Haar wavelets are usually defined as [0,1]. Here we consider the definition interval in general [A, B].

Suppose $M = 2^J$, where J is the highest level of accuracy (here, we mean a natural number that satisfies our demand depending on the accuracy in the coding). Divide the interval [A, B] into 2M sub-intervals of equal lengths. We denote the length of each substrate by Δx¬;

Then, $\Delta x = \frac{B-A}{2M}$‹

In the following, we introduce two other parameters: the variable accuracy parameter, which is $j = 0,1,\dots,J$, and the transfer parameter $k = 0,1,\dots,m-1$ (here, the symbol $m = 2^j$ is introduced Be). The wavelet counter is denoted by $i = m + k + 1$. Here J is an arbitrary natural number.

i Amin Mojak Hare as:

$$h_i(x) = \begin{cases} 1, & x\epsilon[\xi_1(i), \xi_2(i)], \\ -1, & x\epsilon[\xi_2(i), \xi_3(i)], \\ 0, & o.w, \end{cases} \quad (4)$$

Where

$$\xi_1(i) = A + 2k\mu\Delta x, \qquad \xi_2(i) = A + (2k+1)\mu\Delta x, \quad (5)$$

$$\xi_3(i) = A + 2(k+1)\mu\Delta x, \qquad \mu = \frac{M}{m}.$$

The mode $i = 1$ corresponds to a comparison function.

Here:

$$h_1(x) = \begin{cases} 1, & x \in [A, B], \\ 0, & o.w. \end{cases} \qquad (6)$$

Integration of Haar wavelet:

If we show the Haar wave with the symbol h_i (x):

$$\int_A^x h_i(x)dx = \begin{cases} 0, & x < \xi_1(i), x \geq \xi_3(i), \\ x - \xi_1(i), & x \in [\xi_1(i), \xi_2(i)], \\ \xi_3(i) - x, & x \in [\xi_2(i), \xi_3(i)], \end{cases} \qquad (7)$$

To show the correctness of the above relation, we do the following:

$$\int_{\xi_1(i)}^{\xi_2(i)} h_i(x)dx = \int_{\xi_1(i)}^{x} dx = x]_{\xi_1(i)}^{x} = x - \xi_1(i) \ , x \in [\xi_1(i), \xi_2(i)],$$

$$\int_{\xi_2(i)}^{\xi_3(i)} h_i(x)dx = \int_{x}^{\xi_3(i)} -dx = \int_{x}^{\xi_3(i)} dx = x]_{x}^{\xi_3(i)} = \xi_3(i) - x \ , x \in [\xi_2(i), \xi_3(i)],$$

$$\int_{-\infty}^{x} h_i(x)dx = \int_{A}^{x} (0)dx = 0 \ , x < \xi_1(i),$$

$$\int_{x}^{\infty} h_i(x)dx = \int_{A}^{x} (0)dx = 0 \ , x \geq \xi_3(i).$$

Numerical method: In this section, we present a numerical method for solving equation (2). Here we consider two cases:

If $0 < \xi < 1$, then $\cdot n = 1$, and if $1 < \xi < 2$, then $n = 2$.

First case: using the definition of integral, we will have a fraction of Equation (2):

$$\frac{1}{\Gamma(n-\xi)} \int_0^x \frac{y^{(n)}(\tau)d\tau}{(x-\tau)^{1+\xi-n}} = r(x) + b(x)y(x) + \int_0^x W_1(x,\tau)y(\tau)d\tau$$

$$+ \int_0^1 W_2(x,\tau)y(\tau)d\tau . \qquad (8)$$

If ξ is between 0 and 1, then $n = 1$. Thus Equation (8) becomes the following equation:

$$\frac{1}{\Gamma(1-\xi)} \int_0^x \frac{y'(\tau)d\tau}{(x-\tau)^{\xi}} = r(x) + b(x)y(x) + \int_0^x W_1(x,\tau)y(\tau)d\tau$$

$$+ \int_0^1 W_2(x,\tau)y(\tau)d\tau. \qquad (9)$$

Suppose that $y'(x) \epsilon L_2[0,1]$, then $y'(x)$ can be expressed as follows:

$$y'(x) = \sum_{i=1}^{N} \lambda_i \psi_i(x), \qquad (9)$$

Where N is the local number of points, $\psi_i(x)$ are Haar functions, and λ_i are unknown constant coefficients of Haar wavelet. We want to find the unknown coefficients by the Gaussian elimination method.

Now integrating equation (9), we have from 0 to x:

$$\int_0^x y'(x)dx = \int_0^x \sum_{i=1}^{N} \lambda_i \psi_i(x)dx$$

After integration, we have:

$$y(x) - y(0) = \sum_{i=1}^{N} \lambda_i \int_0^x \psi_i(x)dx.$$

For convenience, we put:

$$L_{i,1}(x) = \int_0^x \psi_i(x)dx.$$

By placing $y(x)$ و $y'(x)$ in equation (9),

$$\frac{1}{\Gamma(1-\xi)} \int_0^x \frac{\sum_{i=1}^{N} \lambda_i \psi_i(\tau)d\tau}{(x-\tau)^{\xi}} = r(x) + b(x)\left(y_0 + \sum_{i=1}^{N} \lambda_i L_{i,1}(x)\right) + \int_0^x W_1(x,\tau)(y_0 + \sum_{i=1}^{N} \lambda_i L_{i,1}(\tau))\, d\tau + \int_0^1 W_2(x,\tau)(y_0 + \sum_{i=1}^{N} \lambda_i L_{i,1}(\tau))\, d\tau.$$

Based on the equations above, we have:

$$\frac{1}{\Gamma(1-\xi)} \int_0^x \frac{\sum_{i=1}^{N} \lambda_i \psi_i}{(x-\tau)^{\xi}}\, d\tau = r(x) + b(x)y_0 + \sum_{i=1}^{N} b(x)\lambda_i L_{i,1}(x) + \int_0^x W_1(x,\tau)\left(y_0 + \sum_{i=1}^{N} \lambda_i L_{i,1}(\tau)\right)d\tau + \int_0^1 W_2(x,\tau)\left(y_0 + \sum_{i=1}^{N} \lambda_i L_{i,1}(\tau)\right)d\tau.$$

By moving the unknowns to the left, we will have:

$$\frac{1}{\Gamma(1-\xi)} \int_0^x \frac{\sum_{i=1}^{N} \lambda_i \psi_i(\tau)d\tau}{(x-\tau)^{\xi}} - b(x)\sum_{i=1}^{N} \lambda_i L_{i,1}(x) - \int_0^x W_1(x,\tau)\sum_{i=1}^{N} \lambda_i L_{i,1}(\tau)d\tau - \int_0^1 W_2(x,\tau)\sum_{i=1}^{N} \lambda_i L_{i,1}(\tau)d\tau = r(x) + b(x)y_0 + y_0\int_0^x W_1(x,\tau)d\tau + y_0\int_0^1 W_2(x,\tau)d\tau$$

$$\frac{1}{\Gamma(1-\xi)} \int_0^x \frac{\sum_{i=1}^{N} \lambda_i \psi_i(\tau)d\tau}{(x-\tau)^{\xi}} - \sum_{i=1}^{N} b(x)\,\lambda_i L_{i,1}(x\ \ ) - \sum_{i=1}^{N} \int_0^x w_1(x,\tau)\lambda_i L_{i,1}(\tau)d\tau -$$

$$-\sum_{i=1}^{N} \int_0^1 W_2(x,\tau)\,\lambda_i L_{i,1}(\tau)d\tau = r(x) + b(x)y_0 + y_0\int_0^x W_1(x,\tau)d\tau + y_0\int_0^1 W_2(x,\tau)d\tau$$

Therefore, we have:

$$\sum_{i=1}^{N} \lambda_i \left(\frac{1}{\Gamma(1-\xi)} \int_0^x \frac{\psi_i(\tau)d\tau}{(x-\tau)^{\xi}} - b(x)L_{i,1}(x) - \int_0^x W_1(x,\tau)L_{i,1}(\tau)d\tau - \int_0^1 W_2(x,\tau)L_{i,1}(\tau)d\tau\right)$$

$$= r(x) + b(x)y_0 + y_0\int_0^x W_1(x,\tau)d\tau + y_0\int_0^1 W_2(x,\tau)d\tau.$$

By replacing local points $j = 0, \dots, N-1$، $x_j = \frac{2j+1}{2N}$ in the variable x, the device N will have the following unknown equation N with unknowns $i = 1, \dots, N$ ، λ_i،
Then, we have:

$$\sum_{i=1}^{N} \lambda_i \left(\frac{1}{\Gamma(1-\xi)} \int_0^{x_j} \frac{\psi_i(\tau)d\tau}{(x_j-\tau)^{\xi}} - b(x_j)L_{i,1}(x_j) \right.$$

$$\left. - \int_0^{x_j} W_1(x_j,\tau)L_{i,1}(\tau)d\tau - \int_0^{1} W_2(x_j,\tau)L_{i,1}(\tau)d\tau \right)$$

$$= r(x_j) + b(x_j)y_0 + y_0 \int_0^{x_j} W_1(x_j,\tau)d\tau + y_0 \int_0^{1} W_2(x_j,\tau)d\tau.$$

The above device can be in the form of a matrix MA=B,
where

$$M(j,i) = \frac{1}{\Gamma(1-\xi)} \int_0^{x_j} \frac{\psi_i(\tau)d\tau}{(x_j-\tau)^{\xi}} - b(x_j)L_{i,1}(x_j)$$

$$- \int_0^{x_j} W_1(x_j,\tau)L_{i,1}(x_j)d\tau - \int_0^{1} W_2(x_j,\tau)L_{i,1}(x_j)d\tau,$$

$A = \lambda_i$,
and

$$B = r(x_j) + b(x_j)y_o + y_o \int_0^{x_j} W_1(x_j,\tau)d\tau + y_0 \int_0^{1} W_2(x_j,\tau)d\tau,$$

$i = 1, \dots, N$,
$j = 0, \dots, N-1$.

We use the lepik method to find the elements of the M matrix [13].

Numerical examples:

The capability of the introduced approach is described in this section. To demonstrate the effectiveness of this strategy, the numerical solution is compared to the exact answer and displayed in tables and figures for each example. For each example, the greatest absolute error and the root mean square error are determined. To obtain numerical results, we use MATLAB software.

Example 1. Voltray integral-differential equation:

Consider $y''(x) + \frac{D^{\alpha}}{x}y(x) + \frac{1}{x-1}y(x) = f(x) + \int_0^x k(x,s)y(s)ds$, with the initial condition $y = (0) = 0$ [4]. The functions $k(x,s)$ و $f(x)$ are chosen in such a way that the correct answer for the problem is as $y(x) = x^3(x-1)$.

Therefore:

$$f(x) = x^{11} - \frac{x^6}{30} + \frac{x^5}{20} + x^3 + \frac{24}{\Gamma(4.7)}x^{2.7} + 12x^2 - \frac{6}{\Gamma(3.7)}x^{1.7} - 6x,$$

$$k(x, s) = x - s.$$

A comparison between the exact and approximate answers is performed graphically in Figure 1 for different values of α. In addition, it can be seen from Table 1 that the maximum absolute errors decrease with an increasing number of network points.

Table 1. Absolute error Maximum square root mean error for example, 1

J	$N = 2^{J+1}$	Error M, b	Error M, M, R	Calculation time per seconds
1	2^2	003-e 2.64083	003-e 1.56908	0.002009
2	2^3	003-e 1.67742	004-e 7.18171	0.002024
3	2^4	004-e 9.12254	004-e 3.28514	0.0046611
4	2^5	004-e 3.68413	004-e 1.26008	0.040329
5	2^6	004-e 1.32855	005-e 4.45546	0.108599
6	2^7	005-e 0.453061	005 e-1.50865	0.271642
7	2^8	005-e 0.49708	006-e 4.97646	1.021353
8	2^9	006-e 4.85139	006-e 1.61344	4.059696
9	2^{10}	006-e 1.55198	007-e 5.16608	9.347953

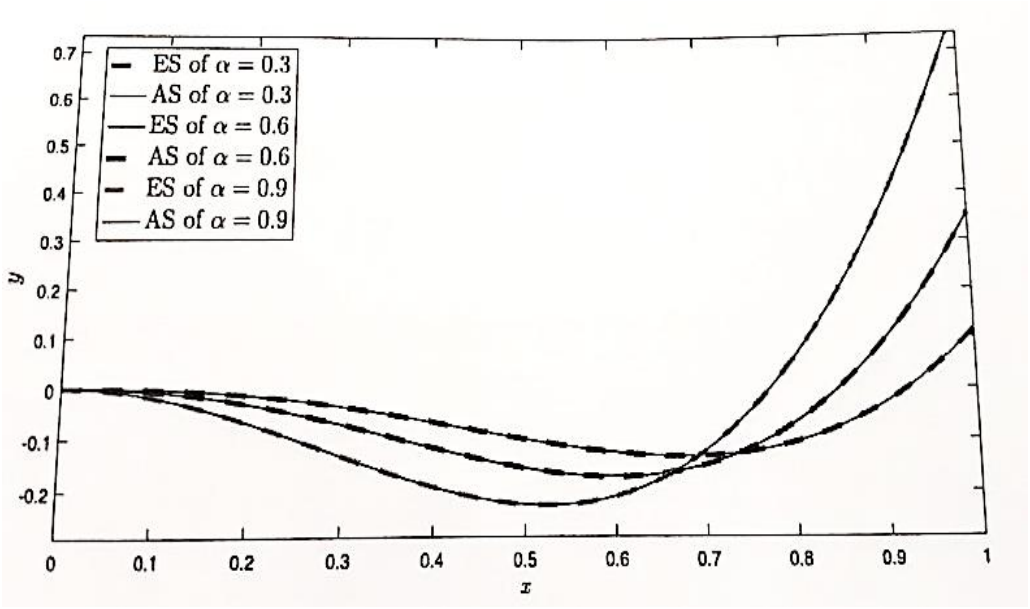

Figure 1. Comparison of approximate and exact answers for different α for 32 local points, for example 1

Example 2-6. The integral-differential equation of the Volta-Fredelm fraction

$$D^{\alpha}y(x) = \frac{\Gamma(4)x^{1.3}}{\Gamma(2.3)} + \frac{\Gamma(3)x^{1.75}}{\Gamma(2.75)} + \frac{\Gamma(4)x^{2.75}}{\Gamma(3.75)} - \sqrt{\Pi}\frac{\Gamma(3)x^{\frac{5}{2}}}{2\Gamma\left(\frac{7}{2}\right)} - \sqrt{\Pi}\frac{\Gamma(4)x^{\frac{7}{2}}}{2\Gamma(\frac{9}{2})}$$

$$+\frac{7x}{36}+\frac{3}{20}+\frac{1}{2}+\int_0^x \frac{y(s)ds}{(x-s)^{\frac{1}{2}}}+\frac{1}{3}\int_0^1 (x-s)y(s)ds,$$

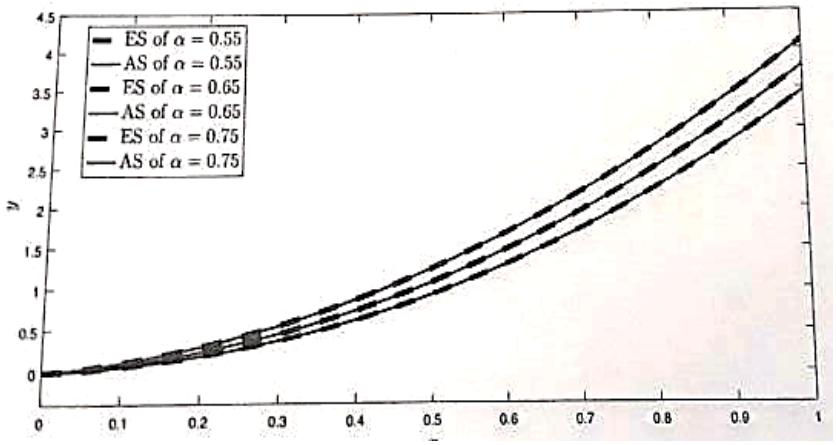

Figure 2 - Comparison of exact and approximate answers for different α for N = 32 Example 2.

Table 2 - Maximum absolute error and mean square root error for example 2

J	$N = 2^{J+1}$	Error M, b	Error M, M, R	Calculation time per seconds
1	2^2	-002e 1.4773	-003e 9.23112	0.001097
2	2^3	-003e 5.78533	-003e 3.47620	0.001490
3	2^4	-003e 2.05252	-003e 1.21423	0.005938
4	2^5	-004e 6.86022	-004e 4.04464	0.017660
5	2^6	-004e 2.21591	-004e 1.30823	0.064919
6	2^7	-005e 7.00770	-005e 4.14922	0.253711
7	2^8	-005e 2.18501	-005e 1.29783	0.011352
8	2^9	-006e 6.74688	-006e 4.01813	4.049050
9	2^{10}	-006e 2.06666	-006e1.23432	10.316231

Let suppose the above table with the initial condition $y(0) = 0$. The exact answer to the problem for $\alpha = 0.25$، is as $y(x) = x^2 + x^3$

It can be seen from Table 2 that the maximum absolute errors decrease with increasing the number of network points.

Example 3: Integral equation - Voltra fraction differential – Fredelm

Consider

$$D^{\alpha}y(x)+\frac{x^2e^x}{5}y(x)=\frac{6x^{(2.25)}}{\Gamma(3.25)}+\int_0^x e^x sy(s)ds+\int_0^1 (4-s^{-3})y(s)ds.$$

With the initial condition:

The exact answer for the $\alpha = 0.75$ is as $y(x) = x^3$

It can be seen from Table 3 that the maximum absolute errors decrease as the number of network points increases.

A comparison between the approximate and exact answers is given graphically in Figure 3 for different values of α.

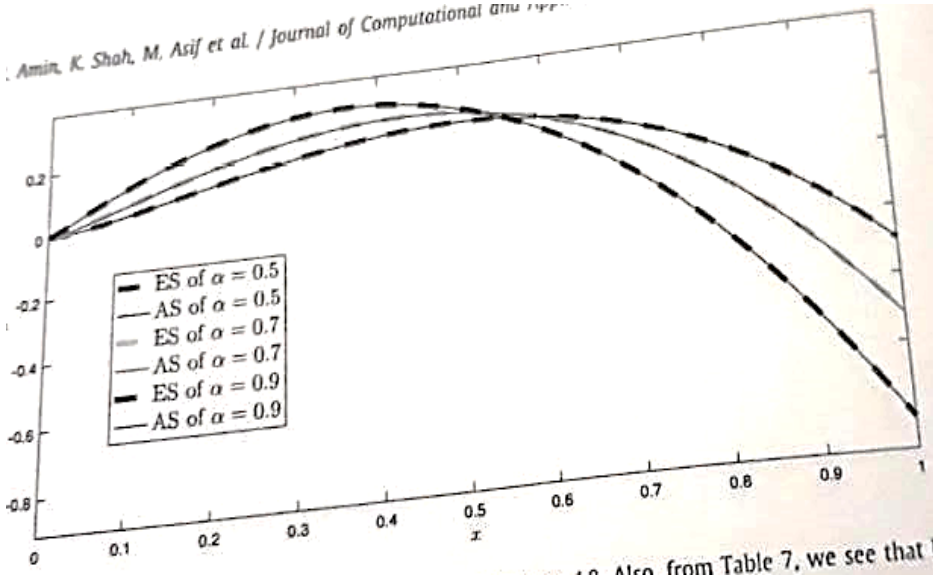

Table 3- Maximum absolute error and root mean square error for example 3

J	$N = 2^{J+1}$	Error M, b	Error M, M, R	Calculation time per seconds
1	2^2	e-0035.89781	0.003e3.96658	0.00970
2	2^3	-0.003e2.91131	-003e1.50961	0.001932
3	2^4	e-0031.51662	-004e6.5542	0.004688
4	2^5	e-0046.42651	-004e2.65591	0.016612
5	2^6	-004e8.99210	-004e1.01443	0.064257
6	2^7	e-0052.14439	-005e3.67230	0.251994
7	2^8	e-0052.95332	-005e1.22286	0.978847
8	2^9	e-0067.14727	-006e3.2958	3.892672
9	2^{10}	-006e1.42274	-006e1.12645	9.763984

Example 4. The model of fractional integral-equation equations covers a large part of real-world problems involving science and engineering. Extensive applications can be found in the field of analysis of these equations.

For example, Kirchhoff's second law states that the net voltage drop across a closed loop is equal to the voltage generated by E (x) (this is essentially applied to energy savings).
The current controlled in the RLC circuit by the fractional integral-differential equation becomes known in

$$D^{\alpha}y(x) = \frac{E(x)}{L} - \frac{R}{L}y(x) - \frac{1}{LC}\int_0^x y(\tau)d\tau\,, \qquad y(0) = y_0,$$

[17], where y is a function of current in terms of time x, R is the resistance, L is the magnetic induction capacity, and C is the capacitance.

Using $y(0) = 0$ و $L = 0.2$ ، $C = 1$ ، $R = 0.4\text{ohm}$ ، $E = 0.2\text{m per } \alpha = 1$

We get the exact answer as follows:

$$y(x) = \frac{1}{2}e^{-x}\sin(2x).$$

A comparison between the approximate and exact answers is made graphically in Figure 4 with different values of α. Also, in Table 4, the values of the absolute error and the mean square root of the problem are given:

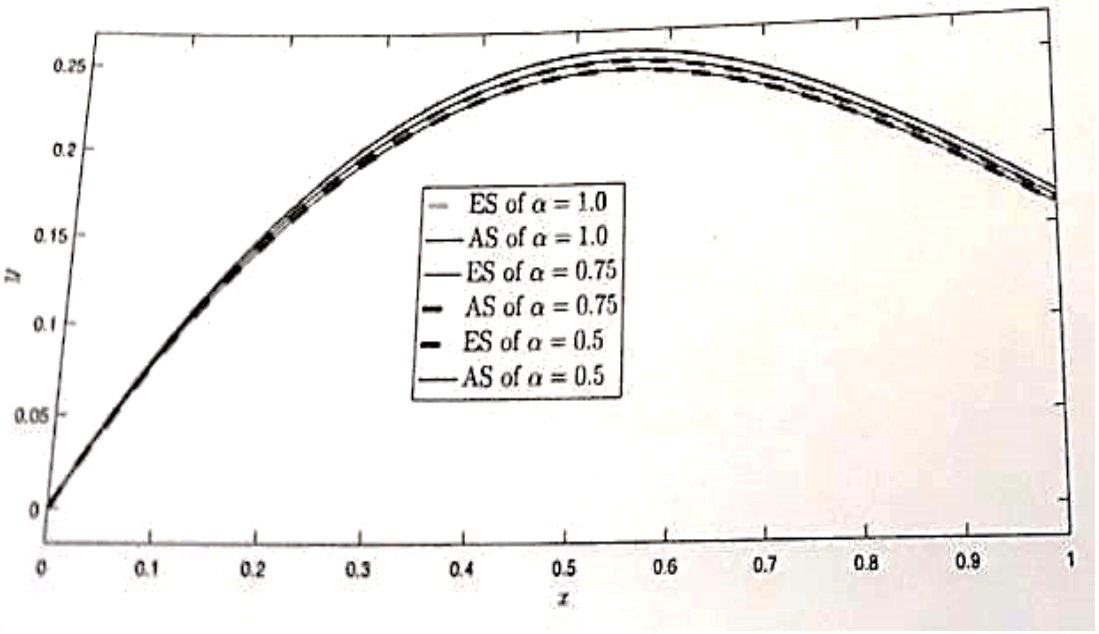

Figure 4. Comparison of the exact and approximate answers for N = 32 and different values of α for example 4

(In the table below "/" is the distinguishing symbol)

Table 4 - Maximum absolute error and mean square root error for example 4

J	$N = 2^{J+1}$	Error M, b	Error M, M, R	Calculation time per seconds
1	2^2	0.0127437	0.0071878	0.001023
2	2^3	0.0035226	0.0017834	0.001554
3	2^4	004e27363004/9	004-e2.44804	0.003243
4	2^5	-004e2.37889	-004e1.11144	0.012620

5	$\mathbf{2^6}$	-005 e 6.02471	-005e2.77825	0.048356
6	$\mathbf{2^7}$	-005e1.51589	-006e6.94539	0.214727
7	$\mathbf{2^8}$	006 e -3.80230	-006e1.73633	0.777766
8	$\mathbf{2^9}$	007e9.52123	-007e4.34082	3.063749
9	$\mathbf{2^{10}}$	007e-2.38224	-007e1.08520	12.310051

Approximate solution of nonlinear fractional integrodifferential equations using Legendre-alternative functions

Legendre functions alternate with fractional order

We define a new set of fractional functions based on alternative Legendre polynomials to obtain the solution of nonlinear fractional integro-differential equations. Alternative legend functions are obtained from fractional order by substituting the variable t with (α> 0) x ^ α in several alternative legend sentences.

To work with $P_{m,i}(x^{\alpha})$, we use the symbol $P^{\alpha}_{m,i}(x)$.

Therefore, we have:

$$P^{\alpha}_{m,i}(x) = \sum_{r=0}^{m-i}(-1)^r \binom{m-i}{r}\binom{m+i+r+1}{m-i} x^{(i+r)\alpha}$$

$$= \sum_{r=i}^{m}(-1)^{r-i} \binom{m-i}{r-i}\binom{m+r+1}{m-i} x^{r\alpha}, \quad i = 0,1,\dots,m. \qquad (10)$$

The set of legendary alternative functions of a fraction relative to the weight function $w(x) = x^{\alpha-1}$ are orthogonal on the [0,1].

$$\int_0^1 P^{\alpha}_{m,k}(x)P^{\alpha}_{m,l}(x)x^{\alpha-1}dx = \frac{1}{(k+l+1)\alpha}\delta_{k,l}, \quad k,l = 0,1\dots,m \qquad (11)$$

- To show the correctness of Equation (11), it is enough to insert the variable change $t = x^{\alpha}$ in Equation (6). Therefore, we have:

$dt = \alpha x^{\alpha-1}dx,$

Determining the limits of the integral: $\begin{cases} t = 0 \Rightarrow x^{\alpha} = 0; \alpha > 0 \xrightarrow{\alpha=\frac{p}{q}} \sqrt[p]{x^q} = 0 \Rightarrow x^q = 0 \Rightarrow x = 0, \\ t = 1 \Rightarrow x^{\alpha} = 1 \xrightarrow{\text{sides to the power } \frac{1}{\alpha}} x = 1, \end{cases}$

$$\alpha\int_0^1 P_{m,k}(x^{\alpha})P_{m,l}(x^{\alpha})x^{\alpha-1}dx = \frac{1}{(k+l+1)}\delta_{k,l},$$

Now, based on the definition: , $P_{m,k}(x^{\alpha}) = P^{\alpha}_{m,k}(x)$ $P_{m,l}(x^{\alpha}) = P^{\alpha}_{m,l}(x)$. Thus, we have:

$$\alpha\int_0^1 P^{\alpha}_{m,l}(x)P^{\alpha}_{m,k}(x)x^{\alpha-1}\,dx = \frac{1}{(k+l+1)}\delta_{k,l} \xrightarrow{\times\frac{1}{\alpha}}$$

$$\Rightarrow \int_0^1 P_{m,l}^{\alpha}(x) P_{m,k}^{\alpha}(x) x^{\alpha-1}\, dx = \frac{1}{(k+l+1)\alpha}\delta_{k,l}.$$

Operational matrix of Legendre functions Alternative order fraction integration:

The main purpose of this section is to obtain the operational matrix of Legendre alternative functions from the fractional integration order. If we take the vector Φ ^ α (x) in relation (16) of the Riemann-Liouville fractional integral operator is obtained from the following equation:

$$I^{\upsilon}\Phi^{\alpha}(x) = F^{(\upsilon,\alpha)}\Phi^{\alpha}(x), \qquad (12)$$

Where $F^{(\upsilon,\alpha)}$ is the operational matrix of fractional integration of order υ from order $(m+1)\times(m+1)$ according to Riemann-Liouville.

Using Equation (10) and the linear property of the Riemann-Liouville fractional integral operator, we obtain for $i = 0,1,\dots,m$:

$$I^{\upsilon}P_{m,i}^{\alpha}(x) = \sum_{r=i}^{m}(-1)^{r-i}\binom{m-i}{r-i}\binom{m+r+1}{m-i} I^{\upsilon}(x^{r\alpha}), \qquad (13)$$

Using the equation $I^{\beta}t^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+\beta+1)}t^{(\beta+\upsilon)}$, $\upsilon > -1$, the above equation results as follows:

$$I^{\upsilon}P_{m,i}^{\alpha}(x) = \sum_{r=i}^{m}(-1)^{r-i}\binom{m-i}{r-i}\binom{m+r+1}{m-i}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+\upsilon+1)}x^{r\alpha+\upsilon} \qquad (14)$$

$$= \sum_{r=i}^{m}\gamma_{mi,r}^{(\upsilon,\alpha)}\, x^{r\alpha+\upsilon},$$

Where

$$\gamma_{mi,r}^{(\upsilon,\alpha)} = (-1)^{r-i}\binom{m-i}{r-i}\binom{m+r+1}{m-i}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+\upsilon+1)}x^{r\alpha+\upsilon}. \qquad (15)$$

Now, if we approximate the function $x^{r\alpha+\upsilon}$ in terms of alternative Legendre functions, we have:

$$x^{r\alpha+\upsilon} \simeq \sum_{j=0}^{m}\delta_{r,j}^{(\upsilon,\alpha)}P_{m,j}^{\alpha}(x). \qquad (16)$$

Substituting Equation (19) into Equation (18) for i = 0,1,…, m, we obtain:

$$I^{\upsilon}P_{m,i}^{\alpha}(x) \simeq \sum_{r=i}^{m}\gamma_{mi,r}^{(\upsilon,\alpha)}\sum_{j=o}^{m}\delta_{r,j}^{(\upsilon,\alpha)}\, P_{m,j}^{\alpha}(x) = \sum_{j=0}^{m}\Big(\sum_{r=i}^{m}\omega_{mi,j,r}^{(\upsilon,\alpha)}\Big)\, P_{m,j}^{\alpha}(x), \qquad (17)$$

Where

$$\omega_{mi,j,r}^{(\upsilon,\alpha)} = \gamma_{mi,r}^{(\upsilon,\alpha)}\delta_{r,j}^{(\upsilon,\alpha)}.$$

Equation (17) can be written as follows:

$$I^{\upsilon}P_{m,i}^{\alpha} \simeq \left[\sum_{r=i}^{m}\ \omega_{mi,0,r}^{(\upsilon,\alpha)}, \sum_{r=i}^{m}\omega_{mi,1,r}^{(\upsilon,\alpha)}, \dots, \sum_{r=i}^{n}\omega_{mi,m,r}^{(\upsilon,\alpha)}\right]\Phi^{\alpha}(x), \quad i = 0,1,\dots,m. \qquad (18)$$

Where

$$\sum_{r=i}^{m}\omega_{mi,0,r}^{(\upsilon,\alpha)} = \left(\sum_{r=i}^{m}\omega_{m0,0,r,}^{(\upsilon,\alpha)}, \dots, \sum_{r=i}^{m}\omega_{mm,0,r}^{(\upsilon,\alpha)}\right)^{T}.$$

$$\sum_{r=i}^{m} \omega_{mi,m,r}^{(\upsilon,\alpha)} = \left(\sum_{r=i}^{m} \omega_{m0,m,r}^{(\upsilon,\alpha)}, \dots, \sum_{r=i}^{m} \omega_{mm,m,r}^{(\upsilon,\alpha)}\right)^{T}.$$

Note that:

$$I^{\upsilon}\Phi^{\alpha}(x) = I^{\upsilon}\left[P_{m,0}^{\alpha}(x), P_{m,1}^{\alpha}(x), P_{m,2}^{\alpha}(x), \dots, P_{m,m}^{\alpha}(x)\right]^{T}$$
$$= \left[I^{\upsilon}P_{m,0}^{\alpha}(x), I^{\upsilon}P_{m,1}^{\alpha}(x), I^{\upsilon}P_{m,2}^{\alpha}(x), \dots, I^{\upsilon}P_{m,m}^{\alpha}(x)\right]^{T}.$$

Therefore (18), we have:

$$I^{\upsilon}\Phi^{\alpha}(x) = \left(\left[\sum_{r=0}^{m} \omega_{m0,0,r}^{(\upsilon,\alpha)}, \sum_{r=0}^{m} \omega_{m0,1,r}^{(\upsilon,\alpha)}, \dots, \sum_{r=0}^{n} \omega_{m0,m,r}^{(\upsilon,\alpha)}\right]\Phi^{\alpha}(x), \dots ,\right.$$
$$\left.\left[\sum_{r=m}^{m} \omega_{mm,0,r}^{(\upsilon,\alpha)}, \sum_{r=m}^{m} \omega_{mm,1,r}^{(\upsilon,\alpha)}, \dots, \sum_{r=m}^{n} \omega_{mm,m,r}^{(\upsilon,\alpha)}\right]\Phi^{\alpha}(x)\right)$$

As a result, we have:

$$I^{\upsilon}\Phi^{\alpha}(x) == \begin{bmatrix} \sum_{r=0}^{m} \omega_{m0,0,r}^{(\upsilon,\alpha)} & \sum_{r=0}^{m} \omega_{m0,1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=0}^{m} \omega_{m0,m,r}^{(\upsilon,\alpha)} \\ \sum_{r=1}^{m} \omega_{m1,0,r}^{(\upsilon,\alpha)} & \sum_{r=1}^{m} \omega_{m1,1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=1}^{m} \omega_{m1,m,r}^{(\upsilon,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=m-1}^{m} \omega_{m(m-1),0,r}^{(\upsilon,\alpha)} & \sum_{r=m-1}^{m} \omega_{m(m-1),1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=m-1}^{m} \omega_{m(m-1),m,r}^{(\upsilon,\alpha)} \\ \omega_{mm,0,m}^{(\upsilon,\alpha)} & \omega_{mm,1,m}^{(\upsilon,\alpha)} & \cdots & \omega_{mm,m,m}^{(\upsilon,\alpha)} \end{bmatrix} \begin{bmatrix} P_{m,0}^{\alpha}(x) \\ \vdots \\ P_{m,m}^{\alpha}(x) \end{bmatrix}$$

Therefore:

$$F^{(\upsilon,\alpha)} = \begin{bmatrix} \sum_{r=0}^{m} \omega_{m0,0,r}^{(\upsilon,\alpha)} & \sum_{r=0}^{m} \omega_{m0,1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=0}^{m} \omega_{m0,m,r}^{(\upsilon,\alpha)} \\ \sum_{r=1}^{m} \omega_{m1,0,r}^{(\upsilon,\alpha)} & \sum_{r=1}^{m} \omega_{m1,1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=1}^{m} \omega_{m1,m,r}^{(\upsilon,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=m-1}^{m} \omega_{m(m-1),0,r}^{(\upsilon,\alpha)} & \sum_{r=m-1}^{m} \omega_{m(m-1),1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=m-1}^{m} \omega_{m(m-1),m,r}^{(\upsilon,\alpha)} \\ \omega_{mm,0,m}^{(\upsilon,\alpha)} & \omega_{mm,1,m}^{(\upsilon,\alpha)} & \cdots & \omega_{mm,m,m}^{(\upsilon,\alpha)} \end{bmatrix} \quad (20)$$

Numerical method

In this section, we present a numerical method for solving the fractional integral-differential problem (1) - (2).

To solve this equation, the first fraction $D^{\alpha}u(x)$in terms of alternative Legendre functions as:

$$D^{\upsilon}u(x) \simeq D^{\upsilon}u_m(x) = C^T\Phi^{\alpha}(x), \qquad (21)$$

Is extended, where $\Phi^{\alpha}(x)$, C is defined in the previous section; now, by applying ^ υ on both sides (26), we will have:

$I^{\upsilon}D^{\upsilon}u(x) \simeq I^{\upsilon}D^{\upsilon}u_m(x) = I^{\upsilon}C^T\Phi^{\alpha}(x)$.

We know:

$$I^{\upsilon}D^{\upsilon}u(x) = u(x) - \sum_{i=0}^{n-1} u^{(i)}(0)\frac{x^i}{i!} \xrightarrow{I^{\upsilon}D^{\upsilon}u(x)=I^{\upsilon}C^T\Phi^{\alpha}(x)} u(x) - \sum_{i=0}^{n-1} u^{(i)}(0)\frac{x^i}{i!} = I^{\upsilon}C^T\Phi^{\alpha}(x)$$

$$= C^T I^{\upsilon}\Phi^{\alpha}(x) \xrightarrow{(24)} C^T F^{(\upsilon,\alpha)}\Phi^{\alpha}(x) \Rightarrow$$

$$\Rightarrow u(x) = \sum_{i=0}^{n-1} u^{(i)}(0)\frac{x^i}{i!} + I^{\upsilon}C^T\Phi^{\alpha}(x) = \sum_{i=0}^{n-1} u^{(i)}(0)\frac{x^i}{i!} + C^T I^{\upsilon}\Phi^{\alpha}(x)$$

$$= \sum_{i=0}^{n-1} u^{(i)}(0)\frac{x^i}{i!} + C^T F^{(\upsilon,\alpha)}\Phi^{\alpha}(x).$$

As a result, according to Equation (2), we have:

$$u_m(x) \simeq u(x) = C^T F^{(\upsilon,\alpha)}\Phi^{\alpha}(x) + \sum_{k=0}^{n-1}\frac{x^k}{k!}\tau_k \,. \qquad (22)$$

It is worth mentioning that $u_m(x)$ is the approximate answer of Equations (1) - (2).

In equation (22) $F^{(\upsilon,\alpha)}$, the operational matrix of fractional integration رت is dependent on the Legendre alternative functions of the fractional-order, which is defined in Equation 20.

Now we have to replace Equations (21) and (22) with (1):

$$C^T\Phi^{\alpha}(x) = F\left(x, C^T F^{(\upsilon,\alpha)}\Phi^{\alpha}(x) + \sum_{k=0}^{n-1}\frac{x^k}{k!}\tau_k, \int_0^x k_1(x,s)G_1\left(s, C^T F^{(\upsilon,\alpha)}\Phi^{\alpha}(s)\right.\right.$$

$$\left.\left. + \sum_{k=0}^{n-1}\frac{s^k}{k!}\tau_k\right) ds, \int_0^1 k_2(x,s)G_2\left(s, C^T F^{(\upsilon,\alpha)}\Phi^{\alpha}(s) + \right.\right.$$

$$\left.\left. + \sum_{k=0}^{n-1}\frac{s^k}{k!}\tau_k\right) ds\right) + Res_m(x). \qquad (23)$$

Where ‹x∈[0,1] ‹$Res_m(x)$is the residual error. This is the error that occurs when substituting the approximate solution in the integrodifferential equation.

Using the Gauss-Legender numerical integration method (reference [18]) to calculate the integral in Equation (23), we obtain:

$$C^T\Phi^{\alpha}(x) = F\left(x, C^T F^{(\upsilon,\alpha)}\Phi^{\alpha}(x) + \sum_{k=0}^{n-1}\frac{x^k}{k!}\tau_k,\right.$$

$$\left.\frac{x}{2}\sum_{j=1}^{\tilde{m}}\omega_j k_1\left(x, \frac{x}{2} + \frac{x}{2}\xi_j\right)G_1\left(\frac{x}{2} + \frac{x}{2}\xi_j, C^T F^{(\upsilon,\alpha)}\Phi^{\alpha}\left(\frac{x}{2} + \frac{x}{2}\xi_j\right)\right.\right.$$

$$+\sum_{k=0}^{n-1}\frac{\left(\frac{x}{2}+\frac{x}{2}\xi_j\right)^k}{k!}\tau_k),\frac{1}{2}\sum_{j=1}^{\tilde{m}}\omega_j k_2\left(x,\frac{1+\xi_j}{2}\right)G_2(\frac{1+\xi_j}{2},$$

$$C^T F^{(\upsilon,\alpha)}\Phi^\alpha(\frac{1+\xi_j}{2})+\sum_{k=0}^{n-1}\frac{\left(\frac{1+\xi_j}{2}\right)^k}{k!}\tau_k))+E_{\tilde{m}}+Res_m(x), \quad (24)$$

Where $j=0,\dots,\widetilde{m}$، ξ_j are the Legendary polynomial roots $P_{\tilde{m}}(x)$, and $E_{\widetilde{m}}$ و $\omega_j=\frac{-2}{(\tilde{m}+1)P'_{\tilde{m}}(\xi_j)P_{(\widetilde{m})+1}(\xi_j)}$ is the error between the Gaussian rule and the exact integral is given [18].

By putting the transferred legendre polynomial roots in equation (24), we have:

$$C^T\Phi^\alpha(x_i)=F(x_i,C^T F^{(\upsilon,\alpha)}\Phi^\alpha(x_i)+\sum_{k=0}^{n-1}\frac{x_i^k}{k!}\tau_k,\frac{x_i}{2}\sum_{j=1}^{\tilde{m}}\omega_j k_1\left(x_i,\frac{x_i}{2}+\frac{x_i}{2}\xi_j\right)$$

$$G_1\left(\frac{x_i}{2}+\frac{x_i}{2}\xi_j,C^T F^{(\upsilon,\alpha)}\Phi^\alpha\left(\frac{x_i}{2}+\frac{x_i}{2}\xi_j\right)+\sum_{k=0}^{n-1}\frac{\left(\frac{x_i}{2}+\frac{x_i}{2}\xi_j\right)^k}{k!}\tau_k\right),$$

$$\frac{1}{2}\sum_{j=1}^{\tilde{m}}\omega_j k_2\left(x_i,\frac{1+\xi_j}{2}\right)G_2(\frac{1+\xi_j}{2},C^T F^{(\upsilon,\alpha)}\Phi^\alpha(\frac{1+\xi_j}{2})+\sum_{k=0}^{n-1}\frac{\left(\frac{1+\xi_j}{2}\right)^k}{k!}\tau_k)). \quad (25)$$

The equations obtained from equation (30) for i = 0,1,…, m are nonlinear equations that can be solved by Newton's iterative method. After solving the recent nonlinear equations of the vector, C is obtained. Then the function u (x) can be obtained from Equation (22):

Convergence analysis

In this convergence section, we review the proposed method for solving fractional differential integral equations.

Theorem 2. Assuming that u and u_m are the approximate and exact answers of Equations (2) - (1), respectively:

$\|U\|<\infty,$ (A)

$K_1=\text{Max}\,|k_1(x,s)|,K_2=\text{Max}\,|k_2(x,s)|;(x,s)\epsilon[0,1]\times[0,1],$ (B)

G_2, G_1, F apply to Lipschitz conditions, respectively, with constants η_2, η_1, (see Chapter 1)

$\Gamma(\upsilon)-\eta-K_1\eta\eta_1-K_2\eta\eta_2\neq 0,$ (C)

Then:

$\|u-u_m\|\to 0.$

Proof: Suppose e_m specifies the error function as

$$e_m(x)=u(x)-u_m(x),$$

If we take D ^ υ from the sides of the above expression, we get the result from (1):

$$D^{\upsilon}e_m(x) = F(x,u(x),\int_0^x k_1(x,s)G_1\big(s,u(s)\big)ds,$$
$$\int_0^1 k_2(x,s)G_2\big(s,u(s)\big)ds)$$
$$-F(x,u_m(x),\int_0^x k_1(x,s)G_1\big(s,u_m(s)\big)ds,$$

$$\int_0^1 k_2(x,s)G_2\big(s,u_m(s)\big)ds) - E_{\tilde{m}} - Res_m(x)\ . \qquad (26)$$

Using the definitions of the fractional integral derivative, if we take the above expression I ^ υ, we will have the rewrite (31) as follows:

$$e_m(x) = I^{\upsilon}(F(x,u(x),\int_0^x k_1(x,s)G_1\big(s,u(s)\big)ds,$$
$$\int_0^1 k_2(x,s)G_2\big(s,u(s)\big)ds)$$
$$-F(x,u_m(x),\int_0^x k_1(x,s)G_1\big(s,u_m(s)\big)ds,$$
$$\int_0^1 k_2(x,s)G_2\big(s,u_m(s)\big)ds) - I^{\upsilon}(E_{\tilde{m}}) - I^{\upsilon}(Res_m(x)), \quad (27)$$

Where $Res_m(x)$، is the residual error, which was described earlier (in the Numerical Methods section).
From (26), it follows:

$$e_m(x) = \Lambda_1(x) - \Lambda_2(x) - \Lambda_3(x), \qquad (27)$$
$$\Lambda_1(x) = I^{\upsilon}(F(x,u(x),\int_0^x k_1(x,s)G_1\big(s,u(s)\big)ds,$$
$$\int_0^1 k_2(x,s)G_2\big(s,u(s)\big)ds)\ ,$$
$$-F(x,u_m(x),\int_0^x k_1(x,s)G_1\big(s,u_m(s)\big)ds)),$$
$$\int_0^1 k_2(x,s)G_2\big(s,u_m(s)\big)ds)),$$
$$\Lambda_2(x) = I^{\upsilon}(E_{\tilde{m}}),$$
$$\Lambda_3(x) = I^{\upsilon}\big(Res_m(x)\big).$$

We now estimate the above three sentences one by one. For $\Lambda_1$we have:

$$|\Lambda_1(x)| = |\frac{1}{\Gamma(\upsilon)}\int_0^x (x-t)^{\upsilon-1}[(F(t,u(t),\int_0^t k_1(t,s)G_1\big(s,u(s)\big)ds,$$

$$\int_0^1 k_2(t,s)G_2\big(s,u(s)\big)ds) - F(t,u_m(t),\int_0^t k_1(t,s)G_1\big(s,u_m(s)\big)ds,$$

$$\int_0^1 k_2(t,s)G_2\big(s,u_m(s)\big)ds))\ ]\ dt| \leq \frac{1}{\Gamma(\upsilon)}\int_0^x |x-t|^{\upsilon-1}\,|(F(t,u(t),\int_0^t k_1(t,s)G_1\big(s,u(s)\big)ds,$$

$$\int_0^1 k_2(t,s)G_2\big(s,u(s)\big)ds)$$

$$-F(t,u_m(t),\int_0^t k_1(t,s)G_1\big(s,u_m(s)\big)ds,$$

$$\int_0^1 k_2(t,s)G_2\big(s,u_m(s)\big)ds))\ |\ dt. \qquad (28)$$

Since $|x-t| \leq 1$ and G_1،F and G_2 is true for Lipschitz, and it is worth noting that, according to Lip Sheets, the expression inside the inequality integral below (30) is greater than the sum of the expressions inside the right-inequality above (28). That's mean:

$$(\eta + k_1\eta\eta_1 + k_2\eta\eta_2)|\ u(t) - u_m(t)| \geq |(F(t,u(t),\int_0^t k_1(t,s)G_1\big(s,u(s)\big)ds,$$

$$\int_0^1 k_2(t,s)G_2\big(s,u(s)\big)ds)$$

$$-F(t,u_m(t),\int_0^t k_1(t,s)G_1\big(s,u_m(s)\big)ds, \qquad (30\ )$$

$$\int_0^1 k_2(t,s)G_2\big(s,u_m(s)\big)ds))\ dt|.$$

Therefore, we have the following inequality:

$$|\Lambda_1(x)| \leq \frac{1}{\Gamma(\upsilon)}\int_0^x (\eta + k_1\eta\eta_1 + k_2\eta\eta_2)|\ u(t) - u_m(t)|dt,$$

From the inequality $0 \leq t \leq x \leq 1$ and the equation $|\ u(t) - u_m(t)| = |\ e_m(t)|$, we have:

$$|\Lambda_1(x)| \leq \frac{1}{\Gamma(\upsilon)}\int_0^1 (\eta + k_1\eta\eta_1 + k_2\eta\eta_2)\ |\ e_m(t)|dt\,.$$

$$\int_0^1 |e_m(x)| = \int_0^1 |\ u(t) - u_m(t)| = \big||e_m|\big|, |e_m(x)| = |\ u(t) - u_m(t)|$$

We have the following inequality:

$$\|\Lambda_1\| \leq \frac{(\eta + k_1\eta\eta_1 + k_2\eta\eta_2)}{\Gamma(\upsilon)}\|e_m\|$$

For Λ_3, Λ_2, the term is calculated as follows:

$$\|\Lambda_2\| \leq \frac{1}{\Gamma(\upsilon)}\|E_{\tilde{m}}\|.$$

$$\|\Lambda_2\| \leq \frac{1}{\Gamma(\upsilon)} \|\mathrm{Res}_m\|.$$

Therefore:

$$\|e_m\| \leq \|\Lambda_1\| + \|\Lambda_2\| + \|\Lambda_3\|.$$

$$\|e_m\| \leq \frac{(\eta + k_1\eta\eta_1 + k_2\eta\eta_2)}{\Gamma(\upsilon)} \|e_m\| + \frac{1}{\Gamma(\upsilon)} \|E_{\tilde{m}}\| + \frac{1}{\Gamma(\upsilon)} \|\mathrm{Res}_m\|,$$

$$\leq \frac{(\eta + k_1\eta\eta_1 + k_2\eta\eta_2)\|e_m\| + \|E_{\tilde{m}}\| + \|\mathrm{Res}_m\|}{\Gamma(\upsilon)},$$

$$\leq \frac{(\eta + k_1\eta\eta_1 + k_2\eta\eta_2)\|e_m\|}{\Gamma(\upsilon)} + \frac{\|E_{\tilde{m}}\| + \|\mathrm{Res}_m\|}{\Gamma(\upsilon)},$$

$$\|e_m\| - \frac{(\eta + k_1\eta\eta_1 + k_2\eta\eta_2)\|e_m\|}{\Gamma(\upsilon)} \leq \frac{\|E_{\tilde{m}}\| + \|\mathrm{Res}_m\|}{\Gamma(\upsilon)},$$

$$\frac{\|e_m\|(\Gamma(\upsilon) - \eta - k_1\eta\eta_1 - k_2\eta\eta_2)}{\Gamma(\upsilon)} \leq \frac{\|E_{\tilde{m}}\| + \|\mathrm{Res}_m\|}{\Gamma(\upsilon)},$$

Now by multiplying the sides of the last phrase in $\frac{\Gamma(\upsilon)}{(\Gamma(\upsilon)-\eta-k_1\eta\eta_1-k_2\eta\eta_2)}$, the following equation results:

$$\|e_m\| \leq \frac{\|E_{\tilde{m}}\| + \|\mathrm{Res}_m\|}{\Gamma(\upsilon) - \eta - k_1\eta\eta_1 - k_2\eta\eta_2}, \quad)$$

It should be noted that the term used in the above relations is the 2-Euclidean term.

Numerical examples:

Example 5 - Suppose the integral-differential equations of the Volta fraction are given in the following figure.

$$\begin{cases} D^{\frac{1}{2}} u(x) = f(x)u(x) + g(x) + \sqrt{x}\int_0^x u^2(s)ds, 0 \leq x \leq 1, \\ u(0) = 0, \end{cases}$$

Where

$$f(x) = 2\sqrt{x} + 2x^{\frac{3}{2}} - \left(\sqrt{x} + x^{\frac{3}{2}}\right) \ln(1 + x),$$

$$g(x) = \frac{2\mathrm{Arcsinh}\sqrt{x}}{\sqrt{\pi}}\sqrt{1 + x.}$$

The exact answer to this question is equal to $u(x) = \mathrm{Ln}(1 + x)$.

Table 5 shows the absolute error obtained between the approximate answers and the exact answer $|u(x) - u_m(x)|$ for $m = 12$. Also, for different values of α in Table 5, the absolute error for different values of m with the method presented with $\alpha = \frac{1}{2}$ is compared with the results of the Jacobi polynomial method given in [19]. In this table, M_1 is the degree of Jacobian polynomials given in reference [19].

Table 5- Absolute errors with $\alpha = 1/2$ for example 5 methods | Absolute error

Method [30]	

$\mathbf{M_1 = 4}$	$\mathbf{1 \times 10^{-2}}$
$\mathbf{M_2 = 6}$	$\mathbf{1 \times 10^{-2}}$
Presented method	
$\mathbf{m = 4}$	$\mathbf{2.68 \times 10^{-2}}$
$\mathbf{m = 6}$	$\mathbf{8.74 \times 10^{-4}}$
$\mathbf{m = 8}$	$\mathbf{9.74 \times 10^{-5}}$
$\mathbf{m = 10}$	$\mathbf{1.21 \times 10^{-5}}$
$\mathbf{m = 12}$	$\mathbf{1.01 \times 10^{-7}}$

Example 6. Suppose the integral-differential equations of the Volta and Fredelm fractions are given below.

$$\begin{cases} D^{2.2}u(x) + \frac{1}{3}\int_0^x (x+s)[u(s)]^2 ds + \frac{1}{4}\int_0^1 (x-s)[u(s)]^3 ds, 0 \le x \le 1, \\ u(0) = u'(0) = u''(0) = 0. \end{cases}$$

Where

$$f(x) = \frac{\Gamma(4)}{\Gamma(1.8)}x^{0.8} - \frac{5}{56}x^8 - \frac{1}{40}x + \frac{1}{44}.$$

The exact answer of the question is $u(x) = x^3$.

To compare the numerical results obtained by the Legender wavelet wavelet method, CAS (LWM) and the proposed method, we want to get help from term l_∞.

l_∞ The absolute error rate is determined by the following equation:

$$\|e_m\| = \max_{0 \le i \le m-1} \{|u(x_i) - u_m(x_i)|\}, \quad x_i = \frac{i}{m}, i = 0,1 \dots, m-1,$$

Where u (x) is the exact answer and u_m (x) is the numerical answer.

Table 6 shows the absolute error l_∞ obtained by wavelet methods LWM, CAS and the proposed method with α = 3.1 and m = 9.12 [20].

Table 6 l_∞ term absolute errors with α = 1/3 for example 6

Presented method			LWM		CAS	
$\mathbf{m = 9}$	$\mathbf{m = 12}$		$\mathbf{m = 16}$	$\mathbf{m = 32}$	$\mathbf{m = 16}$	$\mathbf{m = 32}$
$\mathbf{2.42 \times 10^7}$	$\mathbf{1.25 \times 10^{-3}}$	$\mathbf{2.14 \times 10^{-3}}$	$\mathbf{4.43 \times 10^{-3}}$	$\mathbf{5.24 \times 10^{-3}}$	$\mathbf{7.84 \times 10^{-2}}$	$\mathbf{l_\infty}$
0.219	**0.079**	-	-	-	-	CPU-time

Example - Suppose the integral-differential equations of the mixed Volta and Fredelm fractions are given below.

$$\begin{cases} D^{0.15}u(x) + f(x) + \frac{1}{4}\int_0^x \frac{u(s)}{(x-s)^{\frac{1}{2}}}ds + \frac{1}{7}\int_0^1 e^{x+s}u(s)ds, & 0 \le x \le 1, \\ u(0) = 0, \end{cases}$$

Table 7 . Approximate answer at distinct points with $\alpha = 1$ for example 7

X	LWCM		LWM		Presented method		Exact answer
	$\hat{m} = 8$	$\hat{m} = 16$	$\hat{m} = 8$	$\hat{m} = 16$	$m = 8$	$m = 12$	
0	0.0025	0.001	0.0016	0.0006	-0.0003	-0.0003	0.000
1.8	-0.1007	-0.1023	-0.1014	-0.1058	-0.1079	-0.1084	-0.1094
2.8	-0.17619	-0.1813	-0.1774	-0.1829	-0.1851	-0.1859	-0.1875
3.8	-0.2224	-0.2257	-0.2231	-0.2239	-0.2308	-0.2319	-0.2344
4.8	-0.2431	-0.2442	-0.2382	-0.2447	-0.2455	-0.2470	-0.2500
5.8	-0.2230	-0.2281	-0.2227	-0.2292	-0.2290	-0.2306	-0.2344
6.8	-0.1804	-0.1829	-0.1765	-0.1827	-0.1815	-0.1835	-0.1875
7.8	-0.1011	-0.1055	-0.990	-0.1052	-0.1033	-0.1051	-0.1084
CPU-time	-	-	-	-	0.063	0.235	

Example 8- Consider the population growth model of a fraction of a species. This model is defined by the nonlinear fractional voltage-differential integral equation [21]:

$$(53)\begin{cases} D^{\upsilon}u(x) = au(x) - bu^2(x) - cu(x)\int_0^x u(s)ds, & 0 \le \upsilon \le 1, \\ u(0) = 0.1, \end{cases}$$

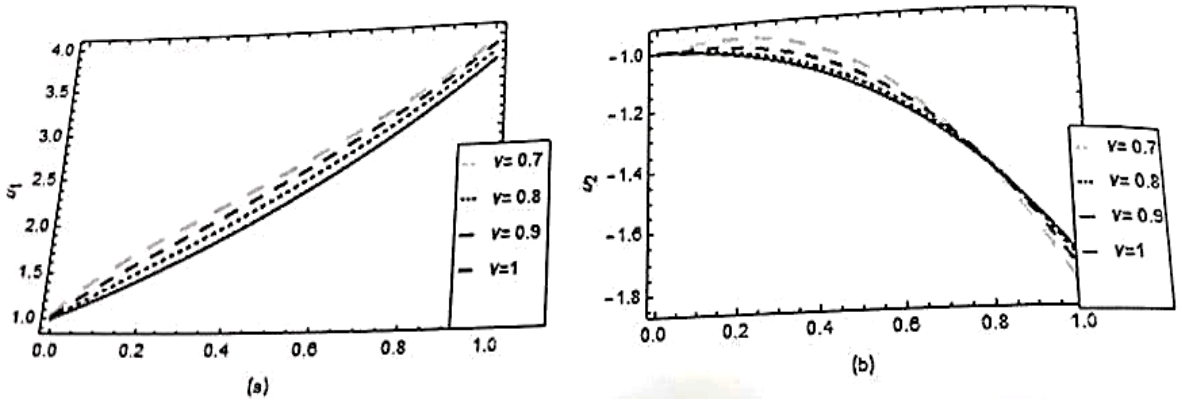

Figure 5-Drawing (a): $u_1(x)$ and)b(: $u_2(x)$ with $m = 5$ and distinct values of υ for example 8

X	Method [33]		Methods	
	$\upsilon = 0.5$	$\upsilon = 0.75$	$\upsilon = 0.5$	$\upsilon = 0.75$
0.1	4.95×10^{-3}	9.53×10^{-3}	2.04×10^{-5}	9.22×10^{-6}
0.2	1.50×10^{-3}	9.22×10^{-4}	9.22×10^{-4}	4.88×10^{-5}
0.3	1.31×10^{-3}	3.34×10^{-3}	9.22×10^{-4}	1.71×10^{-5}
0.4	1.12×10^{-4}	9.04×10^{-4}	9.22×10^{-5}	3.91×10^{-6}

0.5	2.22×10^{-4}	2.77×10^{-5}	9.22×10^{-10}	9.09×10^{-12}
0.6	1.75×10^{-4}	9.22×10^{-4}	9.22×10^{-5}	3.53×10^{-7}
0.7	9.37×10^{-4}	8.13×10^{-4}	9.22×10^{-4}	1.05×10^{-7}
0.8	6.79×10^{-4}	1.42×10^{-4}	9.22×10^{-4}	3.98×10^{-7}
0.9	1.76×10^{-3}	2.76×10^{-3}	7.93×10^{-6}	2.98×10^{-8}

Where u is the population with the same characteristics at time x, a> 0 is the birth rate, b> 0 is the density factor, and c> 0 is the death rate.

Analytical solution of Equation (31) is not available, so to show the efficiency of the proposed method, we define the remaining function as follows:

$$\mathrm{Res}(x) = D^{\upsilon}u_m(x) - bu_m^2(x) + cu_m(x)\frac{x}{2}\sum_{j=1}^{\tilde{m}}\omega_j u_m\left(\frac{x}{2}+\frac{x}{2}\xi_j\right). (31)$$

As an explain, the latter residual function gives an approximate exact answer, which can be exploited experimentally.

Conclusion

To solve the integral-differential equations of the Volta and Volta-Fredelm fractions, the Haar wavelet local point method was devised in this study. The approximate solution converges to a precise result when the tables and figures are compared. As the number of node points grows, so do the absolute maximum errors and root mean squares. As a result, we believe that this method can deliver precise and satisfactory outcomes.

The fractional integration integration matrix $F^{(\upsilon,\alpha)}$ and a general formula for fractional alternative Legendre functions were obtained in this study.

Then, using Legendre alternative functions and their operational matrices, as well as the local point approach, we devised a numerical method. This method is developed for solving nonlinear fractional integral-differential equations numerically. Several examples are given to show that the proposed method for solving fractional integral-differential equations is valid and efficient. The following are some of the benefits of the proposed method:

1. It has been demonstrated that only a limited number of Legendre alternative functions are needed to achieve high success accuracy and satisfactory outcomes.
2. Polysyllabic alternative sentences have one degree of freedom, whereas alternative Legendre functions have two degrees of freedom (m,) from the fractional order in Equation (10). (m).
3. Each member in the set of alternative Legendre polynomials (P_m) is of degree m, but each member in the set of Legendre polynomials is of degree i≤m.
4. We acquire the operational matrix of Legendre alternative functions directly from fractional order, rather than transforming fractional alternative Legendre functions to tiller functions from fractional order. In summary, in this dissertation, we use the Legendre alternative functions of the fraction to solve nonlinear integral-differential fraction equations with beginning conditions, resulting in a new set of defined Legendary alternative functions of the fraction. To answer these equations, we first create the Riemann-Liouville operational matrix, which contains all of these fractional functions as components, and then utilize this matrix with the punctuation approach to arrive at an algebraic device. We will get the desired outcome by

solving the device with MATLAB software and using the resultant equation for an estimated solution. The Haar wavelet is also used to solve the Volta-Fredham and Volta fractions' integro-differential equations. If the Haar technique converts these equations into a system of linear algebraic equations, the Gaussian elimination method is used to solve the problem, and the convergence of these two methods is then evaluated.

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